

DEFECTS IN NEMATIC SHELLS: A Γ -CONVERGENCE DISCRETE-TO-CONTINUUM APPROACH

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ABSTRACT. In this paper we rigorously investigate the emergence of defects on Nematic Shells with genus different from one. This phenomenon is related to a non trivial interplay between the topology of the shell and the alignment of the director field. To this end, we consider a discrete XY system on the shell M , described by a tangent vector field with unit norm sitting at the vertices of a triangulation of the shell. Defects emerge when we let the mesh size of the triangulation go to zero, namely in the discrete-to-continuum limit. In this paper we investigate the discrete-to-continuum limit in terms of Γ -convergence.

1. INTRODUCTION

Liquid Crystals are states of matter with intermediate properties between those of conventional liquids and those of crystalline solids. For instance, in the so-called nematic phase, the constituent molecules — which are typically rod-shaped — are randomly distributed in space but tend to self-align, thus retaining long-range orientational order but no positional order. The peculiar and surprising optical properties of Liquid Crystals are nowadays fundamental in the modern displays for computers and mobile phones.

Liquid Crystals offer many intriguing and fascinating examples of a non trivial interplay between topology, geometry, partial differential equations and physics. Interestingly, Liquid Crystals manifest several visual representations of the underlying geometric constraints. For instance, the word Nematic itself originates from the Greek word $\nu\eta\mu\alpha$ and refers to a particular type of topological defects that these type of Liquid Crystals exhibit.

In this paper, we are interested in exploring these interplays in an even more severe situation, that of *Nematic Shells*. A *Nematic Shell* is a rigid colloidal particle with a typical dimension in the micrometer scale coated with a thin film of nematic liquid crystal whose molecular orientation is subjected to a tangential anchoring. The study of these structures has recently received a good deal of interest. As suggested by Nelson [25], the interest in *Nematic Shells* is related to the possibility of using them as building blocks of mesoatoms with a controllable valence.

From a mathematical point of view, a *Nematic Shell* is usually represented as a two dimensional compact surface M (without boundary, for simplicity) embedded in \mathbb{R}^3 . As it happens for nematic liquid crystals occupying a domain in \mathbb{R}^2 or in \mathbb{R}^3 , the basic mathematical description is given in terms of a unit-norm vector field named director, describing the local orientation of the rod-shaped molecules of the crystal ([34]). When dealing with nematic shells, the local orientation of the molecules described via a unit-norm tangent vector field $\mathbf{n} : M \rightarrow \mathbb{R}^3$ with $\mathbf{n}(x) \in T_x M$ for any $x \in M$, $T_x M$ being the tangent plane at the point x ([33], [20], [25], [23], [24], [29], [30]).

The study of these structures offers a non trivial interplay between the geometry and the topology of the fixed substrate and the tangential anchoring constraint. Indeed, as observed in [35] and [9], the liquid crystal equilibrium (and all its stable configurations, in general) is the

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result of the competition between two driving principles: on the one hand the minimization of the “curvature of the texture” penalized by the elastic energy, and on the other the frustration due to constraints of geometrical and topological nature, imposed by anchoring the nematic to the surface of the underlying particle.

Moreover, the interaction between the local orientation of the molecules and the topology of the surface M (and possibly of the boundary conditions, if any) can induce the formation of topological defects, that is regions of rapid changes in the director field \mathbf{n} . It is important to note that point of defects play the role of hot spots for the formation of the mesoatoms suggested by Nelson in [25], thus understanding the formation and, possibly, the energetics of defect configurations is extremely significant for applications. This type of problems have been already discussed in the physics community, see e.g. [36] and [9] and references therein.

When dealing with smooth vector fields, the classical Poincaré-Hopf Theorem establishes a link between the existence of a continuous tangent vector field with unit norm on a surface M and the topology of the surface itself. To have a clue on what happens for Nematic Shells, let us consider the simplest form of the energy (see [33], [20], [25]):

$$(1.1) \quad E(\mathbf{v}) := \frac{1}{2} \int_M |\mathbf{D}\mathbf{v}|^2 dS,$$

where \mathbf{D} is the covariant derivative on M . It turns out that the rigorous analysis of nematic shells has to face with possible weak forms of the Poincaré-Hopf Theorem. In particular, introducing the “Sobolev set”

$$W_{\tan}^{1,2}(M; \mathbb{S}^2) := \{ \mathbf{v} : M \rightarrow \mathbb{R}^3, \ |\mathbf{v}| = 1, \ \mathbf{v}(x) \in T_x M \text{ for a.a. } x \in M, \ |\mathbf{D}\mathbf{v}| \in L^2(M) \},$$

we have that (see [30] and [12])

$$W_{\tan}^{1,2}(M; \mathbb{S}^2) \neq \emptyset \Leftrightarrow \chi(M) = 0,$$

where $\chi(M)$ is the Euler Characteristic of M . Consequently, the emergence of defects is exactly related to the choice of the topology of M via the Euler characteristic. More in detail (see [12]), the precise relation between the topology of the surface and the topological charge of the defects $x_i \in M$ for $i = 1, \dots, k$ is again given by an extension of the Poincaré-Hopf Theorem to vector fields with Sobolev regularity, namely

$$\sum_{i=1}^k d_i = \chi(M)$$

where d_i is the topological charge of the defect at the point x_i .

The goal of this paper is to understand the emergence of defects for shells of genus different from one (that is, non zero Euler Characteristic). The defect generation is related to the impossibility, for shells with $\chi \neq 0$, of supporting a tangent, unit-norm vector field with the Sobolev regularity above. Thus, a possible strategy would be to relax one of the above constraints, for instance the unit-norm constraint as in the Ginzburg-Landau theory (see, for instance, [8], [15], [26], [17], [27]).

In this paper, we choose another point of view and instead of a continuous model we rather consider a discrete one with the molecules sitting at the vertices of a triangular mesh approximating the surface M . One of the advantages of this approach is that it paves the way for a computational analysis in terms of finite elements.

The model we consider here is a variant of the well-known XY-spin model, which is widely regarded as a prototypical example of a discrete spin system where phase transitions that are mediated by topological defects occur. Such phase transitions were first identified by Kosterlitz and Thouless [19] (also based on previous work by Berezinskii [7]), who were awarded the 2016

Nobel Prize for Physics, together with Haldane, in recognition of their discoveries on topological phases of matter. XY -models have also attracted attention in the mathematical community; see, for instance, [1], [2], [11], where the discrete-to-continuum limit of such models, and their connection with the continuum Ginzburg-Landau theory, is explored. The aforementioned papers are concerned with the study of “flat” situations, i.e. the model is set on a domain $\Omega \subseteq \mathbb{R}^2$; the dynamics for an XY -model on a “curved” torus has been numerically explored e.g. in [31], via a Monte-Carlo approach.

In this paper, we aim to address the mathematical analysis of an XY -model on surfaces. More precisely, given a closed surface $M \subseteq \mathbb{R}^3$ with $\chi(M) \neq 0$, we consider a family of triangulations \mathcal{T}_ε of M with the vertices $i \in \mathcal{T}_\varepsilon^0$ lying on M and with mesh size ε , i.e. $\varepsilon = \max_{T \in \mathcal{T}_\varepsilon} \text{diam}(T)$ (see Section 2 for the details). Any point $i \in \mathcal{T}_\varepsilon^0$ is occupied by a unit-norm tangent vector $\mathbf{v}_\varepsilon(i) \in \mathbf{T}_i M$. Our energy functional takes the form

$$XY_\varepsilon(\mathbf{v}_\varepsilon) := \frac{1}{2} \sum_{i \neq j \in \mathcal{T}_\varepsilon^0} \kappa_\varepsilon^{ij} |\mathbf{v}_\varepsilon(i) - \mathbf{v}_\varepsilon(j)|^2,$$

where the coefficients κ_ε^{ij} are the entries of the stiffness matrix of M , that is, the finite-element discretization of the Laplace-Beltrami operator (see (H₂) for the definition). Configurations with defects emerge when we let $\varepsilon \rightarrow 0$, namely in the discrete-to-continuum limit. Thus, the very goal of this paper is to analyze this limit in terms of Γ -convergence, in the spirit of [1], [2]. Our main result (see Theorem A) exactly relates the emergence of defects with the topology of the shell M . Following the flat case (see [17], [1], [2]), we introduce the so-called vorticity measure $\hat{\mu}_\varepsilon(\mathbf{v}_\varepsilon)$ of a discrete field \mathbf{v}_ε , which is a kind of discrete notion of the Jacobian. This quantity captures all the relevant “topological information” of \mathbf{v}_ε . For sequences $(\mathbf{v}_\varepsilon)_\varepsilon$ that satisfy a logarithmic energy bound (e.g., minimizers), we show that $\hat{\mu}_\varepsilon(\mathbf{v}_\varepsilon)$ converges, in a suitable topology, to a measure of the form $2\pi \sum_{i=1}^k d_i \delta_{x_i} - G dS$, where the points $x_i \in M$ correspond to the position of the defects, the coefficients $d_i \in \mathbb{Z}$ are the topological charges, G is the Gauss curvature and dS is the area element on M . The proof follows the steps of analogous results in the Ginzburg-Landau literature (see, in particular, [17], [15] for the continuous setting and [1], [2] for the discrete XY -setting).

The results in this paper represent the first, albeit important, step in the understanding of the mechanism of defect generation on nematic shells. Among the various open and interesting problems that remain to be addressed, there is the problem of the energetics of defects configurations. In the Ginzburg-Landau theory, this problem has been first addressed, in a rigorous way, by Brezis, Bethuel and Hélein in [8]. The literature on this topic is very vast (see [3] and references therein) and includes also results for the discrete XY model on the plane, see [2]. In a nutshell, the energy one is looking for is the so-called Renormalized Energy. In the “flat situation” this energy originates from the first term in the development with respect to the Γ -convergence. and depends on the position of the singularities. When dealing with a curved substrate, the analysis of [36] suggests that the Renormalized Energy should relate the location of the defects with the curvature of the shell. We will address the rigorous Γ -convergence derivation of the Renormalized Energy in a forthcoming paper.

Even if our analysis was motivated by Nematic Shells, the study of the interplay between the topological properties of the domain and the possible formation of singularities with infinite energy, is common to other models. The issues we aim at addressing have a more general flavor and are independent of the particular system. For instance, the emergence of (topological) defects is ubiquitous in nature (see [18] and references therein). In particular, the system configurations exhibiting defects have often a universal feature, due to the topological origin of defects. Moreover, energy functionals such as (1.1) are commonly used also to model Amphiphilic molecules

exhibiting an hexatic bond orientational order ([22], [10]). Thus, the horizon of our analysis and results is much wider than just Nematic Shells.

2. THE DISCRETE MODEL

In this section we introduce the discrete setting we will use in the rest of the paper. The mathematical analysis of (1.1) bears some analogy with the analysis of harmonic maps (see [30]). In this discrete setting, we use the formalism developed in [6] for the numerics of harmonic maps between manifolds.

2.1. The discrete formalism. Let $M \subseteq \mathbb{R}^3$ be a smooth, compact, connected surface without boundary, oriented by the choice of a smooth, unit normal field $\gamma: M \rightarrow \mathbb{R}^3$. We denote by dS the area element induced by this choice of the orientation and by G the Gauss curvature of M . Let $U \subseteq \mathbb{R}^3$ be an open tubular neighbourhood of M , where the nearest-point projection $P: U \rightarrow M$ is well-defined and smooth.

For any $\varepsilon \in (0, \varepsilon_0]$, we let \mathcal{T}_ε be a triangulation of M , that is, a finite collection of non-degenerate affine triangles $T \subseteq \mathbb{R}^3$ with the following property: the intersection of any two triangles $T, T' \in \mathcal{T}_\varepsilon$ is either empty or a common subsimplex of T, T' . The parameter ε is the mesh size, namely we assume $\varepsilon = \max_{T \in \mathcal{T}_\varepsilon} \text{diam}(T)$. The sets of vertices and edges of \mathcal{T}_ε will be denoted by $\mathcal{T}_\varepsilon^0, \mathcal{T}_\varepsilon^1$, respectively. We will always assume that $\mathcal{T}_\varepsilon^0 \subseteq M$. We set $\widehat{M}_\varepsilon := \cup_{T \in \mathcal{T}_\varepsilon} T$, so \widehat{M}_ε is the piecewise-affine approximation of M induced by \mathcal{T}_ε .

We assume that the family of triangulations $(\mathcal{T}_\varepsilon)$ satisfies the following conditions.

- (H₁) Let $T_{\text{ref}} \subseteq \mathbb{R}^2$ be a reference triangle of vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. There exists a constant $\Lambda > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0]$ and any $T \in \mathcal{T}_\varepsilon$, the (unique) affine bijection $\phi: T_{\text{ref}} \rightarrow T$ satisfies

$$\text{Lip}(\phi) \leq \Lambda\varepsilon, \quad \text{Lip}(\phi^{-1}) \leq \Lambda\varepsilon^{-1}.$$

Here $\text{Lip}(\phi)$ denotes the Lipschitz constant of ϕ , $\text{Lip}(\phi) := \sup_{x \neq y} |x - y|^{-1} |\phi(x) - \phi(y)|$.

- (H₂) For any $\varepsilon \in (0, \varepsilon_0]$ and any $i, j \in \mathcal{T}_\varepsilon^0$ with $i \neq j$, there holds

$$\kappa_\varepsilon^{ij} := - \int_{\widehat{M}_\varepsilon} \nabla \widehat{\varphi}_{\varepsilon,i} \cdot \nabla \widehat{\varphi}_{\varepsilon,j} dS \geq 0,$$

where the hat function $\widehat{\varphi}_{\varepsilon,i}$ is the unique piecewise-affine, continuous function $\widehat{M}_\varepsilon \rightarrow \mathbb{R}$ such that $\widehat{\varphi}_{\varepsilon,i}(j) = \delta_{ij}$ for any $j \in \mathcal{T}_\varepsilon^0$.

- (H₃) For any $\varepsilon \in (0, \varepsilon_0]$, $\widehat{M}_\varepsilon \subseteq U$ and the restriction of the nearest-point projection $\widehat{P}_\varepsilon := P|_{\widehat{M}_\varepsilon}: \widehat{M}_\varepsilon \rightarrow M$ has a Lipschitz inverse. Moreover, we have $\text{Lip}(\widehat{P}_\varepsilon) + \text{Lip}(\widehat{P}_\varepsilon^{-1}) \leq \Lambda$ for some ε -independent constant Λ .

Remark 2.1. (i) (H₁) is equivalent to the following condition: there exists a constant $\Lambda > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0]$ and any triangle $T \in \mathcal{T}_\varepsilon$, there holds

$$\Lambda^{-1}\varepsilon \leq \text{diam}(T) \leq \Lambda\varepsilon \quad \text{and} \quad \alpha_{\min}(T) \geq \Lambda^{-1},$$

where $\alpha_{\min}(T)$ stands for the minimum of the angles of T . Meshes that satisfy this condition are called *quasi-uniform* in the numerical literature. Since the manifold M is compact and smooth, and hence has bounded curvature, $\alpha_{\min}(T) \geq \Lambda^{-1}\varepsilon$ implies that the number of neighbours of a given vertex is uniformly bounded with respect to ε .

- (ii) A sufficient condition for (H₂) is the following: for any pair of triangles $T_1, T_2 \in \mathcal{T}_\varepsilon$ that share a common edge e , let α_i be the angle in T_i opposite to e (for $i \in \{1, 2\}$). If $\alpha_1 + \alpha_2 \leq \pi$ for every edge e as above, then (H₂) holds (see e.g. [6, Lemma 1.4.1]). Triangular meshes that satisfy (H₂) are called *weakly acute*.

- (iii) If \mathcal{T}_ε satisfies (H₂) and if $\widehat{\varphi}, \widehat{\tau} \in C(\widehat{M}_\varepsilon, \mathbb{R})$ are piecewise-affine functions on the triangles of \mathcal{T}_ε , then

$$(2.1) \quad \int_{\widehat{M}_\varepsilon} \nabla \widehat{\varphi} \cdot \nabla \widehat{\tau} \, dS = \sum_{i,j \in \mathcal{T}_\varepsilon^0} \kappa_\varepsilon^{ij} (\widehat{\varphi}(i) - \widehat{\varphi}(j)) (\widehat{\tau}(i) - \widehat{\tau}(j)).$$

- (iv) The condition (H₃) is introduced to rule out pathological examples, and to make sure that \widehat{M}_ε is indeed a good approximation of M . It is not meant to be sharp. There are algorithmic ways to construct triangulations that are quasi-uniform, weakly acute and satisfy (H₃), for instance, Delaunay meshes (see e.g. [32], [4]).

The main characters of our analysis will be unit-norm tangent discrete vector fields on $M \cap \mathcal{T}_\varepsilon^0$, namely maps

$$(2.2) \quad \mathbf{v}_\varepsilon: \mathcal{T}_\varepsilon^0 \rightarrow \mathbb{R}^3 \text{ s.t. } |\mathbf{v}| = 1 \text{ and } \mathbf{v}_\varepsilon(i) \cdot \boldsymbol{\gamma}(i) = 0 \text{ for any } i \in \mathcal{T}_\varepsilon^0.$$

We will denote with $T(\mathcal{T}_\varepsilon; \mathbb{S}^2)$ the space of such discrete vector fields. Given $\mathbf{v}_\varepsilon \in T(\mathcal{T}_\varepsilon; \mathbb{S}^2)$, $\widehat{\mathbf{v}}_\varepsilon: \widehat{M}_\varepsilon \rightarrow \mathbb{R}^3$ denotes the piecewise-affine interpolant of \mathbf{v}_ε . Note that $\widehat{\mathbf{v}}_\varepsilon$ can be represented using the basis functions $\widehat{\varphi}_{\varepsilon,i}$ in this way:

$$(2.3) \quad \widehat{\mathbf{v}}_\varepsilon = \sum_{j \in \mathcal{T}_\varepsilon^0} \mathbf{v}_\varepsilon(j) \widehat{\varphi}_{\varepsilon,j}.$$

In a similar way, $\widehat{\boldsymbol{\gamma}}_\varepsilon$ denotes the piecewise-affine interpolant of $\boldsymbol{\gamma}$ restricted to $\mathcal{T}_\varepsilon^0$.

Remark 2.2. In computational applications, it might be convenient to define the set of discrete vector fields (2.2) using some numerical approximation $\boldsymbol{\gamma}_\varepsilon$ of $\boldsymbol{\gamma}$, instead of $\boldsymbol{\gamma}$ itself. The arguments in this paper could easily be adapted to cover this case as well, provided that the approximation $\boldsymbol{\gamma}_\varepsilon$ satisfies an a priori bound such as

$$\sup_{i \in \mathcal{T}_\varepsilon^0} |\boldsymbol{\gamma}_\varepsilon(i) - \boldsymbol{\gamma}(i)| \leq C\varepsilon.$$

2.2. The discrete energy and main results. Given a discrete field $\mathbf{v}_\varepsilon \in T(\mathcal{T}_\varepsilon, \mathbb{S}^2)$, we consider the discrete XY-energy

$$(XY_\varepsilon) \quad XY_\varepsilon(\mathbf{v}_\varepsilon) := \frac{1}{2} \sum_{i \neq j \in \mathcal{T}_\varepsilon^0} \kappa_\varepsilon^{ij} |\mathbf{v}_\varepsilon(i) - \mathbf{v}_\varepsilon(j)|^2.$$

Because the support of the hat function $\widehat{\varphi}_{\varepsilon,i}$ only intersects the triangles that are adjacent to i , we have $\kappa_\varepsilon^{ij} = 0$ if the vertices i, j are not adjacent. Hence, the XY-energy is indeed defined by a nearest-neighbours interaction. Moreover, due to (2.3) we have

$$(2.4) \quad XY_\varepsilon(\mathbf{v}_\varepsilon) = \frac{1}{2} \int_{\widehat{M}_\varepsilon} |\nabla \widehat{\mathbf{v}}_\varepsilon|^2 \, dS.$$

Now, we briefly introduce the important notion of discrete vorticity measure. This measure will be a kind of discrete notion of jacobian for the discrete vector field \mathbf{v}_ε in $T(\mathcal{T}_\varepsilon; \mathbb{S}^2)$. As it happens for the discrete flat case and for Ginzburg Landau case, the vorticity measure of the sequence \mathbf{v}_ε will provide all the informations regarding the emergence of the defects in the $\varepsilon \searrow 0$ limit. Even if we will precisely introduce this measure in the next Subsection 3.4, we briefly present it now for the sake of clarity. Given a triangle $T \in \mathcal{T}_\varepsilon$ we let (i_0, i_1, i_2) be the

vertices of T , sorted in counter-clockwise order with respect to the orientation induced by γ , and let $i_3 := i_0$. For any triangle T , $\hat{\mu}_\varepsilon(\mathbf{v}_\varepsilon) \llcorner T$ is supported on the barycenter of T and

$$(2.5) \quad \hat{\mu}_\varepsilon(\mathbf{v}_\varepsilon)[T] := \sum_{k=0}^2 \left(\frac{\gamma(i_k) + \gamma(i_{k+1})}{2}, \mathbf{v}_\varepsilon(i_k) \times \mathbf{v}_\varepsilon(i_{k+1}) \right).$$

In the limit $\varepsilon \rightarrow 0$, the appearance of defects is related to a measure concentrated on a finite number of points $\{x_1, \dots, x_k\}$ in M . We will denote by X the set of measures on M of the form

$$\mu = \sum_{i=1}^k d_i \delta_{x_i},$$

where $k \in \mathbb{N}$, $d_i \in \mathbb{Z}$ are such that $\sum_i d_i = \chi(N)$ and $x_i \in M$ for $i \in \{1, \dots, k\}$. The space X will be endowed with the topology of flat convergence, that is, the topology induced by the dual norm of Lipschitz functions.

Here is the precise statement of the main result of the paper.

Theorem A. *Suppose that the assumptions (H₁), (H₂) and (H₃) are satisfied. Then, the following results hold.*

(i) Compactness. *If (\mathbf{v}_ε) is a sequence in $T(\mathcal{T}_\varepsilon; \mathbb{S}^2)$ that satisfies the energy bound*

$$(H) \quad XY_\varepsilon(\mathbf{v}_\varepsilon) \leq \Lambda |\log \varepsilon|$$

then, up to subsequences, $\hat{\mu}_\varepsilon(\mathbf{v}_\varepsilon) \xrightarrow{\text{flat}} 2\pi\mu - \text{GdS}$ for some $\mu \in X$.

(ii) Γ -liminf inequality. *Let (\mathbf{v}_ε) be a sequence in $T(\mathcal{T}_\varepsilon; \mathbb{S}^2)$ such that $\hat{\mu}_\varepsilon(\mathbf{v}_\varepsilon) \xrightarrow{\text{flat}} 2\pi\mu - \text{GdS}$ for some $\mu \in X$. Then, there holds*

$$\liminf_{\varepsilon \rightarrow 0} \frac{XY_\varepsilon(\mathbf{v}_\varepsilon)}{|\log \varepsilon|} \geq \pi|\mu|(M).$$

(iii) Γ -limsup inequality. *For any $\mu \in X$ there exists a sequence (\mathbf{v}_ε) in $T(\mathcal{T}_\varepsilon; \mathbb{S}^2)$ such that $\hat{\mu}_\varepsilon(\mathbf{v}_\varepsilon) \xrightarrow{\text{flat}} 2\pi\mu - \text{GdS}$ and*

$$\limsup_{\varepsilon \rightarrow 0} \frac{XY_\varepsilon(\mathbf{v}_\varepsilon)}{|\log \varepsilon|} \leq \pi|\mu|(M).$$

This Theorem will be proved in the next Section 4 (see Proposition 4.8). In this Proposition we will actually prove a slightly stronger result, where the Γ -liminf inequality (ii) is replaced by (a local version of)

$$XY_\varepsilon(\mathbf{v}_\varepsilon) \geq \pi|\mu|(M) |\log \varepsilon| - C.$$

3. PRELIMINARIES

3.1. The metric distortion tensor. By the assumption (H₃), the restriction of the nearest-point projection $\hat{P}_\varepsilon: \widehat{M}_\varepsilon \rightarrow M$ has a Lipschitz inverse $\hat{P}_\varepsilon^{-1}: M \rightarrow \widehat{M}_\varepsilon$. Following [14], we use this pair of maps to compare M with its polyhedral approximation \widehat{M}_ε . For any $x \in M$ such that $\hat{P}_\varepsilon^{-1}(x)$ falls in the interior of a triangle of \widehat{M}_ε (so that \hat{P}_ε^{-1} is smooth in a neighbourhood of x), we let $\mathbf{A}_\varepsilon(x)$ be the unique linear operator $T_x M \rightarrow T_x M$ that satisfies

$$(3.1) \quad (\mathbf{A}_\varepsilon(x)\mathbf{X}, \mathbf{Y}) = \left(d\hat{P}_\varepsilon^{-1}(x)[\mathbf{X}], d\hat{P}_\varepsilon^{-1}(x)[\mathbf{Y}] \right)$$

for any $\mathbf{X}, \mathbf{Y} \in T_x M$. This defines (almost everywhere) a $(1, 1)$ -tensor field $\mathbf{A}_\varepsilon \in L^\infty(M; TM \otimes T^*M)$, which is called *metric distortion tensor* in the terminology of [14]. The metric distortion

tensor is symmetric and positive definite, since the right-hand side of (3.1) is. We introduce a norm $\|\cdot\|_{L^\infty(M)}$ on $L^\infty(M; TM \otimes T^*M)$ by

$$\|\mathbf{A}\|_{L^\infty(M)} := \operatorname{ess\,sup}_{x \in M} \|\mathbf{A}(x)\|_{TM \otimes T^*M},$$

where $\|\cdot\|_{TM \otimes T^*M}$ is the operator norm.

Lemma 3.1. *Suppose that $(\mathcal{T}_\varepsilon)$ satisfies (H_1) and (H_3) . Then, there holds*

$$\|\mathbf{A}_\varepsilon - \operatorname{Id}\|_{L^\infty(M)} + \|\mathbf{A}_\varepsilon^{-1} - \operatorname{Id}\|_{L^\infty(M)} \leq C\varepsilon.$$

Proof. Let $\widehat{\boldsymbol{\nu}}_\varepsilon: \widehat{M}_\varepsilon \rightarrow \mathbb{R}^3$ be a unit normal field to \widehat{M}_ε , which is well defined (and constant) in the interior of each triangle. The assumption (H_1) implies that

$$\|\widehat{\boldsymbol{\nu}}_\varepsilon \circ \widehat{P}_\varepsilon^{-1} - \boldsymbol{\gamma}\|_{L^\infty(M)} \leq C\varepsilon, \quad \|\operatorname{dist}(\cdot, \widehat{M}_\varepsilon)\|_{L^\infty(M)} \leq C\varepsilon$$

for some ε -independent constant C . (One can write M as a smooth graph locally around a point $x \in M$, then use a Taylor expansion; the constant C can be chosen uniformly with respect to x , by a compactness argument.) Thanks to [14, Theorem 1], which gives a formula for \mathbf{A}_ε in terms of $(\widehat{\boldsymbol{\nu}}_\varepsilon \circ \widehat{P}_\varepsilon^{-1}, \boldsymbol{\gamma})$ and $\operatorname{dist}(\cdot, \widehat{M}_\varepsilon)$, we deduce

$$(3.2) \quad \|\mathbf{A}_\varepsilon - \operatorname{Id}\|_{L^\infty(M)} \leq C\varepsilon.$$

Now, the definition (3.1) of \mathbf{A}_ε , together with the fact that $\widehat{P}_\varepsilon^{-1}$ has a Lipschitz inverse \widehat{P}_ε and $\operatorname{Lip}(\widehat{P}_\varepsilon) \leq \Lambda$ by (H_3) , imply that

$$|\mathbf{A}_\varepsilon(x)\mathbf{X}| \geq C|\mathbf{X}|$$

for some constant $C = C(\Lambda)$, a.e. $x \in M$ and all $\mathbf{X} \in T_x M$, whence $\|\mathbf{A}_\varepsilon^{-1}\|_{L^\infty(M)} \leq C$. Thus, we have

$$\|\mathbf{A}_\varepsilon^{-1} - \operatorname{Id}\|_{L^\infty(M)} \leq \|\mathbf{A}^{-1}\|_{L^\infty(M)} \|\operatorname{Id} - \mathbf{A}_\varepsilon\|_{L^\infty(M)} \stackrel{(3.2)}{\leq} C\varepsilon. \quad \square$$

Let $g_\varepsilon \in L^\infty(M; T^*M^{\otimes 2})$ be the metric on M defined by $g_\varepsilon(\mathbf{X}, \mathbf{Y}) := (\mathbf{A}_\varepsilon \mathbf{X}, \mathbf{Y})$, for any smooth fields \mathbf{X} and \mathbf{Y} on M . Given a function $u \in W^{1,2}(M)$, one can define the Sobolev $W^{1,2}$ -seminorm of u with respect to g_ε , i.e.

$$(3.3) \quad |u|_{W_\varepsilon^{1,2}(M)}^2 := \int_M (\mathbf{A}_\varepsilon^{-1} \nabla u, \nabla u) (\det \mathbf{A}_\varepsilon)^{1/2} dS,$$

where ∇ denotes the Riemannian gradient and dS the volume form on M (with respect to the metric induced by \mathbb{R}^3). By construction (3.1), the map $\widehat{P}_\varepsilon^{-1}$ is an isometry between M , equipped with the metric g_ε , and \widehat{M}_ε , with the metric induced by \mathbb{R}^3 . Therefore, given $v \in W^{1,2}(\widehat{M}_\varepsilon; \mathbb{R})$ and a Borel set $U \subseteq M$, there holds

$$(3.4) \quad |v \circ \widehat{P}_\varepsilon^{-1}|_{W_\varepsilon^{1,2}(U)}^2 = \int_{\widehat{P}_\varepsilon^{-1}(U)} |\nabla v|^2 dS.$$

Arguing component-wise, we see that the same equality holds for a (not necessarily tangent) vector field $\mathbf{v}: \widehat{M}_\varepsilon \rightarrow \mathbb{R}^3$ in place of v , provided that we interpret ∇ as a surface gradient (as opposed to the covariant derivative).

3.2. Interpolants of discrete fields. Using assumption (H_3) , to any discrete vector field $\mathbf{v}_\varepsilon \in T(\mathcal{T}_\varepsilon; \mathbb{S}^2)$ we can associate a continuous field $\mathbf{w}_\varepsilon: M \rightarrow \mathbb{R}^3$ by setting $\mathbf{w}_\varepsilon := \widehat{\mathbf{v}}_\varepsilon \circ \widehat{P}_\varepsilon^{-1}$, where $\widehat{\mathbf{v}}_\varepsilon: \widehat{M}_\varepsilon \rightarrow \mathbb{R}^3$ is the affine interpolant of \mathbf{v}_ε . The field \mathbf{w}_ε is Lipschitz-continuous and satisfies $\mathbf{w}_\varepsilon = \mathbf{v}_\varepsilon$ on $\mathcal{T}_\varepsilon^0$, but it is not tangent to M nor unit-valued, in general. However, one can still prove some useful properties.

Lemma 3.2. *Suppose that (H_1) , (H_2) , (H_3) are satisfied. Then, for any $\varepsilon \in (0, \varepsilon_0]$ and any discrete field $\mathbf{v}_\varepsilon \in T(\mathcal{T}_\varepsilon; \mathbb{S}^2)$, \mathbf{w}_ε is Lipschitz-continuous with Lipschitz constant*

$$(3.5) \quad \text{Lip}(\mathbf{w}_\varepsilon) \leq C\varepsilon^{-1}$$

and there holds

$$(3.6) \quad XY_\varepsilon(\mathbf{v}_\varepsilon) = \frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(M)}^2.$$

Proof. From the very definition of $\mathbf{w}_\varepsilon := \widehat{\mathbf{v}}_\varepsilon \circ \widehat{P}_\varepsilon^{-1}$, it follows that

$$\text{Lip}(\mathbf{w}_\varepsilon) \stackrel{(H_3)}{\leq} C \text{Lip}(\widehat{\mathbf{v}}_\varepsilon) \leq C \sup_{[i,j] \in \mathcal{T}_\varepsilon^1} \frac{|\mathbf{v}_\varepsilon(i) - \mathbf{v}_\varepsilon(j)|}{|i - j|} \stackrel{(H_1)}{\leq} C\varepsilon^{-1},$$

To prove (3.6), it is enough to combine (2.1) with (3.4). \square

Lemma 3.3. *Suppose that $(\mathcal{T}_\varepsilon)$ satisfies (H_1) , (H_3) . Then, there exists a constant C such that, for any $\varepsilon \in (0, \varepsilon_0]$ and any $\mathbf{v}_\varepsilon \in T(\mathcal{T}_\varepsilon; \mathbb{S}^2)$, there holds*

$$\|(\mathbf{w}_\varepsilon, \gamma)\|_{L^\infty(M)} \leq C\varepsilon.$$

Proof. Every point $x \in \widehat{M}_\varepsilon$ can be written in the form $x = \lambda_0 i_0 + \lambda_1 i_1 + \lambda_2 i_2$, where $i_k \in \mathcal{T}_\varepsilon^0$ and $\lambda_k \geq 0$, $\lambda_0 + \lambda_1 + \lambda_2 = 1$. Using the definition of the affine interpolant, and the fact that $(\mathbf{v}_\varepsilon(i_k), \gamma(i_k)) = 0$, we can write

$$\begin{aligned} \left| \left(\widehat{\mathbf{v}}_\varepsilon(x), (\gamma \circ \widehat{P}_\varepsilon)(x) \right) \right| &\leq \sum_{k=0}^2 \lambda_k \left| \left(\mathbf{v}_\varepsilon(i_k), (\gamma \circ \widehat{P}_\varepsilon)(x) - (\gamma \circ \widehat{P}_\varepsilon)(i_k) \right) \right| \\ &\leq \|\nabla(\gamma \circ P)\|_{L^\infty(U)} \sup_{T \in \mathcal{T}_\varepsilon} \text{diam}(T). \end{aligned}$$

Thus, using the smoothness of γ and the assumptions (H_1) , (H_3) , we deduce

$$\|(\mathbf{w}_\varepsilon, \gamma)\|_{L^\infty(M)} = \left\| \left(\widehat{\mathbf{v}}_\varepsilon, \gamma \circ \widehat{P}_\varepsilon \right) \right\|_{L^\infty(\widehat{M}_\varepsilon)} \leq C\varepsilon. \quad \square$$

The following property is well-known in the flat case (see e.g. [1, Lemma 2]).

Lemma 3.4. *Suppose that (H_1) is satisfied. Then, there exists a positive constant C such that, for any $0 < \varepsilon \leq \varepsilon_0$ and any discrete field $\mathbf{v}_\varepsilon \in T(\mathcal{T}_\varepsilon; \mathbb{S}^2)$, there holds*

$$\frac{1}{\varepsilon^2} (1 - |\mathbf{w}_\varepsilon|^2)^2 \leq C |\nabla \mathbf{w}_\varepsilon|^2 \quad \text{pointwise on } M.$$

Proof. Thanks to (H_3) , it suffices to show that

$$(3.7) \quad \frac{1}{\varepsilon^2} (1 - |\widehat{\mathbf{v}}_\varepsilon|^2)^2 \leq C |\nabla \widehat{\mathbf{v}}_\varepsilon|^2 \quad \text{on } \widehat{M}_\varepsilon.$$

Let $T \in \mathcal{T}_\varepsilon$ be a triangle with vertices i_0, i_1, i_2 . Any point $x \in T$ can be written as $x = i_0 + \lambda_1(i_1 - i_0) + \lambda_2(i_2 - i_0)$, where λ_1, λ_2 are positive numbers such that $\lambda_1 + \lambda_2 \leq 1$. Using the definition of affine interpolant and that $|\mathbf{v}_\varepsilon(i_0)| = 1$, we obtain that

$$1 - |\widehat{\mathbf{v}}_\varepsilon(x)| \leq |\widehat{\mathbf{v}}_\varepsilon(x) - \mathbf{v}_\varepsilon(i_0)| \leq \sum_{k=1}^2 \lambda_k |\mathbf{v}_\varepsilon(i_k) - \mathbf{v}_\varepsilon(i_0)|,$$

whence, using that $|\widehat{\mathbf{v}}_\varepsilon| \leq 1$ and that $\nabla \widehat{\mathbf{v}}_\varepsilon$ is constant on T , we deduce

$$(1 - |\widehat{\mathbf{v}}_\varepsilon(x)|^2)^2 \leq 4(1 - |\widehat{\mathbf{v}}_\varepsilon(x)|)^2 \leq 8 \sum_{k=1}^2 |\mathbf{v}_\varepsilon(i_k) - \mathbf{v}_\varepsilon(i_0)|^2 = 8 \sum_{k=1}^2 |\nabla \widehat{\mathbf{v}}_\varepsilon(x)(i_k - i_0)|^2.$$

Now, (3.7) follows because $|i_k - i_0| \leq C\varepsilon$, due to (H_1) . \square

As a consequence of Lemmas 3.2 and 3.4, if both (H_1) and (H_2) are satisfied, then we have

$$(3.8) \quad \frac{1}{\varepsilon^2} \int_{\widehat{M}_\varepsilon} (1 - |\mathbf{w}_\varepsilon|^2)^2 \leq CXY_\varepsilon(\mathbf{v}_\varepsilon).$$

3.3. Jacobians of continuous vector fields. In this section, we define the Jacobian determinant of a vector field in the sense of distributions, and we recall a few useful properties. This notion was introduced in the context of Ginzburg-Landau functionals (see e.g. [16]) and in non-linear elasticity (see e.g. [5], [21]). As we are dealing with vector fields over a manifold, it will be useful to recast the theory in the language of differential forms.

Given a map $\mathbf{u} \in (W^{1,1} \cap L^\infty)(M; \mathbb{R}^3)$, we define the “pre-jacobian” or vorticity of \mathbf{u} as the 1-form

$$(3.9) \quad j(\mathbf{u}) := (\gamma, \mathbf{u} \wedge d\mathbf{u}).$$

More explicitly, $j(\mathbf{u})$ is defined via its action on a smooth, *tangent* field \mathbf{w} on M :

$$(3.10) \quad \langle j(\mathbf{u}), \mathbf{w} \rangle = (\gamma, \mathbf{u} \times \nabla_{\mathbf{w}} \mathbf{u}).$$

Here, ∇ stands for the Euclidean gradient in \mathbb{R}^3 , but we can equivalently replace $\nabla_{\mathbf{w}} \mathbf{u}$ with the covariant derivative $D_{\mathbf{w}} \mathbf{u}$ since the scalar product in (3.10) does not depend on the component of $\nabla_{\mathbf{w}} \mathbf{u}$ in the direction of γ .

Suppose now that $\mathbf{u} \in W^{1,1}(M; \mathbb{R}^3)$ is a *unit, tangent* field on M (that is, $|\mathbf{u}| = 1$ and $\mathbf{u} \cdot \gamma = 0$ a.e.), and let $(\mathbf{e}_1, \mathbf{e}_2)$ be a local orthonormal basis for the tangent frame of M . Then, we have

$$(3.11) \quad |j(\mathbf{u})|^2 = \sum_{k=1}^2 |\mathbf{u} \times D_{\mathbf{e}_k} \mathbf{u}|^2 = \sum_{k=1}^2 |D_{\mathbf{e}_k} \mathbf{u}|^2 = |D\mathbf{u}|^2,$$

where we denote by $|\cdot|$ both the norm on the tangent space and the induced norm on the cotangent space. Moreover, we can write locally that

$$(3.12) \quad \mathbf{u} = (\cos \alpha) \mathbf{e}_1 + (\sin \alpha) \mathbf{e}_2$$

for some scalar function α with bounded variation (this follows, e.g., by [13]). A *formal* computation shows that

$$(3.13) \quad j(\mathbf{u}) = d\alpha - \mathbf{A}$$

where \mathbf{A} , called *spin connection*, is the 1-form defined by $\langle \mathbf{A}, \mathbf{w} \rangle := \mathbf{e}_1 \cdot \nabla_{\mathbf{w}} \mathbf{e}_2$. Note that \mathbf{A} depends on the choice of the frame, but its differential is an intrinsic quantity:

$$(3.14) \quad d\mathbf{A} = G dS,$$

where we recall that G is the Gauss curvature of M .

The differential $dj(\mathbf{u})$ will play an important rôle. Since $dj(\mathbf{u})$ is a 2-form, it can be written uniquely as $dj(\mathbf{u}) = f dS$ where $f \in \mathcal{D}'(M)$ is scalar and dS is the volume form on M ; we use the notation $\star dj(\mathbf{u}) := f$. In case $M = \mathbb{R}^2$ (embedded as the xy -plane in \mathbb{R}^3 , so that $\gamma = \mathbf{e}_3$) and \mathbf{u} is a smooth vector field $\mathbb{R}^2 \rightarrow \mathbb{R}^2$, we easily compute that

$$\star dj(\mathbf{u}) = 2 \det \nabla \mathbf{u},$$

thus $\star d\mathbf{j}(\mathbf{u})$ can be thought as a generalization of the Jacobian determinant of \mathbf{u} , up to a constant factor 2. If \mathbf{u} is a unit, tangent field on M , then by differentiating (3.13) we see that $\star d\mathbf{j}(\mathbf{u})$ contains topological information about the singularities of \mathbf{u} . We denote by $\text{ind}(\mathbf{u}, x_i)$ the local degree of \mathbf{u} at the point x_i , that is, the winding number of \mathbf{u} around the boundary of a small disk centered at x_i (see e.g. [12] for more details).

Lemma 3.5. *Let $\mathbf{u} \in W_{\text{tan}}^{1,1}(M; \mathbb{S}^2)$ be a unit tangent field that is smooth, except at a finite number of points x_1, \dots, x_p . Then*

$$\star d\mathbf{j}(\mathbf{u}) = 2\pi \sum_{i=1}^p \text{ind}(\mathbf{u}, x_i) \delta_{x_i} - G \quad \text{in } \mathcal{D}'(M).$$

Proof. For a fixed i , take a test function $\varphi \in C^\infty(M)$ whose support is simply connected and does not contain any singularity of \mathbf{v} other than x_i . Suppose that an orthonormal tangent frame $(\mathbf{e}_1, \mathbf{e}_2)$ is defined on the support of φ . Then, we can locally define an angular variable α which satisfies Equation (3.12) and is smooth, except for a jump across a smooth ray starting at the point x_i . The size of the jump is constant along the ray, and equal to $2\pi \text{ind}(\mathbf{u}, x_i)$. The Lebesgue-absolutely continuous part $d^{\text{ac}}\alpha$ of the differential $d\alpha$ is actually continuous across the jump, and satisfies (analogously to (3.13))

$$(3.15) \quad \mathbf{j}(\mathbf{u}) = d^{\text{ac}}\alpha - \mathbf{A}$$

on the support of φ . Thanks to (3.15), (3.11) and the fact that $\mathbf{u} \in W^{1,1}$, we deduce that $d^{\text{ac}}\alpha \in L^1$.

Now, we compute $\star d(d^{\text{ac}}\alpha)$ in the sense of distributions. For any $\delta > 0$, we have

$$(3.16) \quad \begin{aligned} -\langle d^{\text{ac}}\alpha, \star d\varphi \rangle_{L^2(M \setminus B_\delta(x_i))} &= \int_{M \setminus B_\delta(x_i)} d^{\text{ac}}\alpha \wedge d\varphi = - \int_{M \setminus B_\delta(x_i)} d(d^{\text{ac}}\alpha \wedge \varphi) \\ &= \int_{\partial B_\delta(x_i)} d^{\text{ac}}\alpha \wedge \varphi. \end{aligned}$$

On the other hand, we have

$$(3.17) \quad \left| \int_{\partial B_\delta(x_i)} d^{\text{ac}}\alpha \wedge (\varphi - \varphi(0)) \right| \leq \delta \|\nabla \varphi\|_{L^\infty(M)} \int_{\partial B_\delta(x_i)} |d^{\text{ac}}\alpha| \, ds.$$

We claim that the right-hand side of (3.17) converges to 0 at least along a subsequence $\delta_j \searrow 0$. For otherwise, there would exist positive numbers η and δ_0 such that

$$\delta \int_{\partial B_\delta(x_i)} |d^{\text{ac}}\alpha| \, ds \geq \eta$$

for any $0 < \delta \leq \delta_0$. Dividing by δ both sides of this inequality and integrating over $\delta \in (0, \delta_0)$, we would obtain

$$\int_{B_{\delta_0}(x_i)} |d^{\text{ac}}\alpha| \, dS \geq \eta \int_0^{\delta_0} \frac{d\delta}{\delta} = +\infty,$$

which is impossible because $d^{\text{ac}}\alpha \in L^1$. Then, we find a subsequence $\delta_j \searrow 0$ along which the right-hand side of (3.17) converges to 0. Taking the limit in (3.16) along this subsequence, and using (3.17), we obtain

$$-\langle d^{\text{ac}}\alpha, \star d\varphi \rangle_{L^2(M)} = \lim_{j \rightarrow +\infty} \int_{\partial B_{\delta_j}(x_i)} d^{\text{ac}}\alpha \wedge \varphi(0) = 2\pi \text{ind}(\mathbf{u}, x_i) \varphi(0).$$

Since the operator $\star d$ is $L^2(M)$ -anti-symmetric, the left-hand side of this identity can be interpreted as the duality pairing $\langle \star d(d^{\text{ac}}\alpha), \varphi \rangle$, in the sense of distributions. Combining this with (3.14) and (3.15), the lemma follows. \square

We define a piecewise-continuous counterpart of j . Take a bounded, piecewise-smooth (but not necessarily tangent) map $\mathbf{u}: \widehat{M}_\varepsilon \rightarrow \mathbb{R}^3$ such that, for any edge $e = [i, j] \in \mathcal{T}_\varepsilon^1$,

$$(3.18) \quad \nabla_{j-i} \mathbf{u} = \nabla \mathbf{u}(j - i) \text{ is continuous across } e.$$

For example, the affine interpolant $\mathbf{u} = \widehat{\mathbf{v}}_\varepsilon$ of a discrete field $\mathbf{v} \in T(\mathcal{T}_\varepsilon; \mathbb{S}^2)$ satisfies (3.18). We let

$$(3.19) \quad \widehat{j}_\varepsilon(\mathbf{u}) := (\widehat{\gamma}_\varepsilon, \mathbf{u} \wedge d\mathbf{u}),$$

that is the piecewise-smooth 1-form on \widehat{M}_ε satisfying

$$\langle \widehat{j}_\varepsilon(\mathbf{u}), \mathbf{w} \rangle = (\widehat{\gamma}_\varepsilon, \mathbf{u} \times \nabla_{\mathbf{w}} \mathbf{u})$$

for any piecewise-smooth tangent field \mathbf{w} on \widehat{M}_ε . This form is well-defined and continuous on each triangle of \widehat{M}_ε . Note that $\widehat{j}_\varepsilon(\mathbf{u})$ may not be continuous across an edge $e = [i, j]$ but $\langle \widehat{j}_\varepsilon(\mathbf{u}), i - j \rangle$ is, therefore the integral of $\widehat{j}_\varepsilon(\mathbf{u})$ along e is defined unambiguously.

3.4. Jacobians of discrete vector-fields. We want to define a notion of “jacobian” for a discrete field $\mathbf{v}_\varepsilon \in T(\mathcal{T}_\varepsilon; \mathbb{S}^2)$ and we have two possibilities: either we apply $d\widehat{j}_\varepsilon$ to the affine interpolant $\widehat{\mathbf{v}}_\varepsilon$, or we compute $dj(\mathbf{u}_\varepsilon)$ for a field $\mathbf{u}_\varepsilon: M \rightarrow \mathbb{R}^3$ that interpolates \mathbf{v}_ε . The first possibility corresponds to the measure

$$(3.20) \quad \widehat{\mu}_\varepsilon(\mathbf{v}_\varepsilon) := \sum_{T \in \mathcal{T}_\varepsilon} \left(\int_T d\widehat{j}_\varepsilon(\widehat{\mathbf{v}}_\varepsilon) \right) \delta_{x_T},$$

where δ_{x_T} is the Dirac delta measure supported by the barycentre x_T of T . Let (i_0, i_1, i_2) be the vertices of a triangle $T \in \mathcal{T}_\varepsilon$, sorted in counter-clockwise order with respect to the orientation induced by γ , and let $i_3 := i_0$. Using Stokes’ theorem and the definition of the affine interpolant, we compute

$$(3.21) \quad \begin{aligned} \widehat{\mu}_\varepsilon(\mathbf{v}_\varepsilon)[T] &= \sum_{k=0}^2 \int_{[i_k, i_{k+1}]} (\widehat{\gamma}_\varepsilon, \widehat{\mathbf{v}}_\varepsilon \times \nabla_{i_{k+1}-i_k} \widehat{\mathbf{v}}_\varepsilon) ds \\ &= \sum_{k=0}^2 \left(\frac{\gamma(i_k) + \gamma(i_{k+1})}{2}, \mathbf{v}_\varepsilon(i_k) \times \mathbf{v}_\varepsilon(i_{k+1}) \right). \end{aligned}$$

As for the second approach, we construct a suitable field \mathbf{u}_ε in the following way. We fix a sequence $(t_\varepsilon)_{\varepsilon>0}$ such that

$$(3.22) \quad \frac{\varepsilon |\log \varepsilon|}{t_\varepsilon} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

e.g. $t_\varepsilon := \varepsilon |\log \varepsilon|^2$. Now, reminding that $\mathbf{w}_\varepsilon := \widehat{\mathbf{v}}_\varepsilon \circ \widehat{P}_\varepsilon^{-1}$, for $x \in M$ we define

$$(3.23) \quad \tilde{\mathbf{u}}_\varepsilon(x) := \text{proj}_{T_{xM}} \mathbf{w}_\varepsilon(x) \quad \text{and} \quad \mathbf{u}_\varepsilon(x) := \eta_\varepsilon(|\tilde{\mathbf{u}}_\varepsilon(x)|) \tilde{\mathbf{u}}_\varepsilon(x),$$

where $\eta_\varepsilon(s) := \min\{t_\varepsilon^{-1}, s^{-1}\}$. Note that \mathbf{u}_ε is a Lipschitz tangent field on M and $\mathbf{u}_\varepsilon = \mathbf{v}_\varepsilon$ on $\mathcal{T}_\varepsilon^0$. Next, we set

$$(3.24) \quad \mu_\varepsilon(\mathbf{v}_\varepsilon) := \sum_{T \in \mathcal{T}_\varepsilon} \left(\int_{P(T)} dj(\mathbf{u}_\varepsilon) \right) \delta_{P(x_T)}.$$

Given a Borel set $E \subseteq M$, let E_ε be the union of all the $P(T)$'s such that $T \in \mathcal{T}_\varepsilon$, $P(x_T) \in E$. If $|\tilde{\mathbf{u}}_\varepsilon| \geq 1/4$ on ∂E_ε , then we can find a unit tangent field $\mathbf{U}_\varepsilon \in W_{\tan}^{1,1}(E_\varepsilon; \mathbb{S}^2)$ such that $\mathbf{U}_\varepsilon = \mathbf{u}_\varepsilon$ on ∂E_ε and \mathbf{U}_ε is smooth except at finitely many points. (One can modify $\tilde{\mathbf{u}}_\varepsilon$ in such a way that it is smooth and has 0 as a regular value, then define $\mathbf{U}_\varepsilon := \tilde{\mathbf{u}}_\varepsilon/|\tilde{\mathbf{u}}_\varepsilon|$.) Since $\mu_\varepsilon(\mathbf{v}_\varepsilon)[E] = \int_{E_\varepsilon} d\mathbf{j}(\mathbf{u}_\varepsilon)$ and, by Stokes' theorem, the latter only depends on the restriction of \mathbf{u}_ε to ∂E_ε , we have $\mu(\mathbf{v}_\varepsilon)[E] = \int_{E_\varepsilon} d\mathbf{j}(\mathbf{U}_\varepsilon)$ and hence, by Lemma 3.5,

$$(3.25) \quad \mu_\varepsilon(\mathbf{v}_\varepsilon)[E] = 2\pi \operatorname{ind}(\mathbf{u}_\varepsilon, \partial E_\varepsilon) - \int_{E_\varepsilon} G \, dS.$$

In this sense, the measure $\mu_\varepsilon(\mathbf{v}_\varepsilon)$ can be thought as a generalization of the discrete vorticity defined in [2] (see in particular [2, Remark 2.1]), and immediately provides information on the “topological” behaviour of \mathbf{v}_ε . On the other hand, the measure $\hat{\mu}_\varepsilon(\mathbf{v}_\varepsilon)$ has the advantage of being simpler to evaluate, thanks to (3.21). Luckily, if the XY-energy of the field \mathbf{v}_ε satisfies a logarithmic bound, then the two measures are close to each other.

Proposition 3.6. *Suppose that (H_1) , (H_2) , (H_3) are satisfied. Let $(\mathbf{v}_\varepsilon)_{0 < \varepsilon \leq \varepsilon_0}$ be a sequence of discrete fields that satisfies (H) for some ε -independent constant Λ and any $0 < \varepsilon \leq \varepsilon_0$. Then, there holds*

$$\|\hat{\mu}_\varepsilon(\mathbf{v}_\varepsilon) - \mu_\varepsilon(\mathbf{v}_\varepsilon)\|_{\text{flat}} \leq C \left(\frac{|\varepsilon| \log |\varepsilon|}{t_\varepsilon} + |\varepsilon| \log |\varepsilon| \right).$$

In particular, the difference between the two measures converges to zero in the flat norm as $\varepsilon \rightarrow 0$, if we assume that (3.22) holds. The rest of this section is devoted to the proof of Proposition 3.6. The key fact is the following continuity property for the Jacobian, which is well-known for maps $\mathbf{u}: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ (see e.g. [3, Lemma 2.1]).

Lemma 3.7. *Let \mathbf{u}, \mathbf{w} be (not necessarily tangent) fields in $W^{1,2}(M; \mathbb{R}^3)$. Then, there holds*

$$(3.26) \quad \|\star d\mathbf{j}(\mathbf{u})\|_{L^1(M)} \leq C \left(\|\mathbf{u}\|_{L^2(M)}^2 + \|\nabla \mathbf{u}\|_{L^2(M)}^2 \right),$$

$$(3.27) \quad \|\star d\mathbf{j}(\mathbf{u}) - \star d\mathbf{j}(\mathbf{w})\|_{\text{flat}} \leq \|\mathbf{u} - \mathbf{w}\|_{L^2(M)} \left(\|\nabla \mathbf{u}\|_{L^2(M)} + \|\nabla \mathbf{w}\|_{L^2(M)} \right).$$

Proof. By a density argument, we can assume WLOG that \mathbf{u}, \mathbf{w} are smooth. Using Einstein convention, we can write

$$\mathbf{j}(\mathbf{u}) = \gamma^i j_i(\mathbf{u}), \quad \text{where} \quad j_i(\mathbf{u}) := \epsilon_{ijk} \mathbf{u}^j d\mathbf{u}^k$$

and ϵ_{ijk} is the Levi-Civita symbol, given by $\epsilon_{ijk} := 1$ if (i, j, k) is an even permutation of $(1, 2, 3)$, $\epsilon := -1$ if it is an odd permutation, and $\epsilon_{ijk} := 0$ otherwise. By differentiating, we deduce

$$(3.28) \quad d\mathbf{j}(\mathbf{u}) = \epsilon_{ijk} \mathbf{u}^j d\gamma^i \wedge d\mathbf{u}^k + \epsilon_{ijk} \gamma^i d\mathbf{u}^j \wedge d\mathbf{u}^k,$$

whence (3.26) immediately follows by applying the Hölder inequality and using that $|\nabla \gamma|$ is bounded. We now prove (3.27). A straightforward computation shows that

$$j_3(\mathbf{u}) - j_3(\mathbf{w}) = \frac{1}{2} \left(j_3(\mathbf{u}^1 - \mathbf{w}^1, \mathbf{u}^2 + \mathbf{w}^2) - j_3(\mathbf{u}^2 - \mathbf{w}^2, \mathbf{u}^1 + \mathbf{w}^1) \right)$$

and similar equalities hold for j_1, j_2 , therefore

$$(3.29) \quad d\mathbf{j}(\mathbf{u}) - d\mathbf{j}(\mathbf{w}) = \frac{\epsilon_{ijk}}{2} d \left(\gamma^i j_i(\mathbf{u}^j - \mathbf{w}^j, \mathbf{u}^k + \mathbf{w}^k) \right).$$

Now, fix a function $\varphi \in C_c^\infty(U)$. Thanks to (3.29) and an integration by parts, we deduce

$$\langle \star d\mathbf{j}(\mathbf{u}) - \star d\mathbf{j}(\mathbf{w}), \varphi \rangle = -\frac{\epsilon_{ijk}}{2} \langle \gamma^i j_i(\mathbf{u}^j - \mathbf{w}^j, \mathbf{u}^k + \mathbf{w}^k), \star d\varphi \rangle.$$

The definition of j_i and the Hölder inequality immediately imply

$$\langle \star d j(\mathbf{u}) - \star d j(\mathbf{w}), \varphi \rangle \leq \|\mathbf{u} - \mathbf{w}\|_{L^2(M)} \|\nabla \mathbf{u} - \nabla \mathbf{w}\|_{L^2(M)} \|\nabla \varphi\|_{L^\infty(M)},$$

whence (3.27) follows by taking the supremum over φ . \square

Lemma 3.7 has a counterpart in the piecewise-continuous setting. For further reference, here we only mention that

Lemma 3.8. *Let $\mathbf{u}: \widehat{M}_\varepsilon \rightarrow \mathbb{R}^3$ be a (not necessarily tangent) piecewise-smooth field that satisfies (3.18). Then, there holds*

$$\|\star d \widehat{j}_\varepsilon(\mathbf{u})\|_{L^1(T)} \leq C \left(\|\mathbf{u}\|_{L^2(T)}^2 + \|\nabla \mathbf{u}\|_{L^2(T)}^2 \right) \quad \text{for any } T \in \mathcal{T}_\varepsilon.$$

Proof. We argue as in Lemma 3.7, using that the functions $\widehat{\gamma}_\varepsilon$ are Lipschitz continuous and $\|\nabla \widehat{\gamma}_\varepsilon\|_{L^\infty(\widehat{M}_\varepsilon)} \leq \|\nabla \gamma\|_{L^\infty(M)}$. \square

Proof of Proposition 3.6. Given a piecewise-smooth map $\mathbf{u}: \widehat{M}_\varepsilon \rightarrow \mathbb{R}^3$, we let $j(\mathbf{u}) := (\gamma \circ P, \mathbf{u} \wedge d\mathbf{u})$, i.e. we extend the operator $\mathbf{u} \mapsto j(\mathbf{u})$ to fields \mathbf{u} that are not defined on M by pre-composing γ with the projection $P: U \rightarrow M$. When \mathbf{u} is a piecewise-smooth field, we denote by $d j(\mathbf{u})$, $d \widehat{j}_\varepsilon(\mathbf{u})$ the Lebesgue-absolutely continuous part of the distributional differential of $j(\mathbf{u})$, $\widehat{j}_\varepsilon(\mathbf{u})$ respectively — that is, we neglect any jumps that may arise at the boundary of the regions where \mathbf{u} is smooth.

Let (\mathbf{v}_ε) be a sequence of discrete fields satisfying the logarithmic energy bound (H). The assumption (H) together with (2.4), (3.8) and the fact that $|\widehat{\mathbf{v}}_\varepsilon| \leq 1$ implies that

$$(3.30) \quad \|\widehat{\mathbf{v}}_\varepsilon\|_{L^2(\widehat{M}_\varepsilon)}^2 + \|\nabla \widehat{\mathbf{v}}_\varepsilon\|_{L^2(\widehat{M}_\varepsilon)}^2 + \varepsilon^{-2} \|1 - |\widehat{\mathbf{v}}_\varepsilon|^2\|_{L^2(\widehat{M}_\varepsilon)}^2 \leq C |\log \varepsilon|$$

for any $0 < \varepsilon \leq \varepsilon_0$ and some constant $C = C(M, \Lambda, \varepsilon_0)$, provided that $\varepsilon_0 < 1$. We decompose the difference $\widehat{\mu}_\varepsilon(\mathbf{v}_\varepsilon) - \mu_\varepsilon(\mathbf{v}_\varepsilon)$ as a sum of several terms:

$$\begin{aligned} \widehat{\mu}_\varepsilon(\mathbf{v}_\varepsilon) - \mu_\varepsilon(\mathbf{v}_\varepsilon) &= \underbrace{\widehat{\mu}_\varepsilon(\mathbf{v}_\varepsilon) - \star d \widehat{j}_\varepsilon(\widehat{\mathbf{v}}_\varepsilon)}_{=: A_1} + \underbrace{\star d \widehat{j}_\varepsilon(\widehat{\mathbf{v}}_\varepsilon) - \star d j(\widehat{\mathbf{v}}_\varepsilon)}_{=: A_2} + \underbrace{\star d j(\widehat{\mathbf{v}}_\varepsilon) - \star d j(\mathbf{w}_\varepsilon)}_{=: A_3} \\ &\quad + \underbrace{\star d j(\mathbf{w}_\varepsilon) - \star d j(\tilde{\mathbf{u}}_\varepsilon)}_{=: A_4} + \underbrace{\star d j(\tilde{\mathbf{u}}_\varepsilon) - \star d j(\mathbf{u}_\varepsilon)}_{=: A_5} + \underbrace{\star d j(\mathbf{u}_\varepsilon) - \mu_\varepsilon(\mathbf{u}_\varepsilon)}_{=: A_6}. \end{aligned}$$

Throughout the rest of the proof, we let $\varphi \in C_c^\infty(U)$ be an arbitrarily fixed test function.

Analysis of A_1 . There holds

$$\begin{aligned} \langle \widehat{\mu}_\varepsilon(\mathbf{v}_\varepsilon) - \star d \widehat{j}_\varepsilon(\widehat{\mathbf{v}}_\varepsilon), \varphi \rangle &= \sum_{T \in \mathcal{T}_\varepsilon} \int_T (\varphi(x_T) - \varphi) d \widehat{j}_\varepsilon(\widehat{\mathbf{v}}_\varepsilon) \\ &\leq \|\star d \widehat{j}_\varepsilon(\widehat{\mathbf{v}}_\varepsilon)\|_{L^1(\widehat{M}_\varepsilon)} \|\nabla \varphi\|_{L^\infty(U)} \sup_{T \in \mathcal{T}_\varepsilon} \text{diam}(T) \end{aligned}$$

Using Lemma 3.8, the assumption (H₁) and (3.30), we deduce

$$(3.31) \quad \|\widehat{\mu}_\varepsilon(\mathbf{v}_\varepsilon) - \star d \widehat{j}_\varepsilon(\widehat{\mathbf{v}}_\varepsilon)\|_{\text{flat}} \leq C \varepsilon |\log \varepsilon|.$$

Analysis of A_2 . By integrating by parts on each triangle of the triangulation, we can write

$$\langle \star d \widehat{j}_\varepsilon(\widehat{\mathbf{v}}_\varepsilon) - \star d j(\widehat{\mathbf{v}}_\varepsilon), \varphi \rangle = - \sum_{T \in \mathcal{T}_\varepsilon} \int_T (\widehat{j}_\varepsilon(\widehat{\mathbf{v}}_\varepsilon) - j(\widehat{\mathbf{v}}_\varepsilon)) \wedge (\star d \varphi).$$

The total contribution of the boundary terms vanishes, because each edge appears in the sum twice, with opposite orientations. Then, by the Hölder inequality, we obtain

$$\|\star d \widehat{j}_\varepsilon(\widehat{\mathbf{v}}_\varepsilon) - \star d j(\widehat{\mathbf{v}}_\varepsilon)\|_{\text{flat}} \leq \|\widehat{\gamma}_\varepsilon - \gamma \circ P\|_{L^\infty(\widehat{M}_\varepsilon)} \|\widehat{\mathbf{v}}_\varepsilon\|_{L^2(\widehat{M}_\varepsilon)} \|\nabla \widehat{\mathbf{v}}_\varepsilon\|_{L^2(\widehat{M}_\varepsilon)}.$$

Using that $|\widehat{\mathbf{v}}_\varepsilon| \leq 1$, that $\|\widehat{\gamma}_\varepsilon - \gamma \circ P\|_{L^\infty(\widehat{M}_\varepsilon)} \leq C\varepsilon$ (as a consequence of (H₁)) and (3.30), we conclude that

$$(3.32) \quad \|\star d\widehat{j}_\varepsilon(\widehat{\mathbf{v}}_\varepsilon) - \star d j(\widehat{\mathbf{v}}_\varepsilon)\|_{\text{flat}} \leq C\varepsilon |\log \varepsilon|^{1/2}.$$

Analysis of A_3 . Set $\omega := j(\mathbf{w}_\varepsilon)$, so ω is a 1-form on M and $\widehat{P}_\varepsilon^*(\omega) = j(\widehat{\mathbf{v}}_\varepsilon)$. Since the sets $\widehat{P}_\varepsilon(T)$ for $T \in \mathcal{T}_\varepsilon$ define a Borel partition of M up to sets of measure zero, we can write

$$\langle \star d j(\mathbf{w}_\varepsilon) - \star d j(\widehat{\mathbf{v}}_\varepsilon), \varphi \rangle = \sum_{T \in \mathcal{T}_\varepsilon} \left(\int_{\widehat{P}_\varepsilon(T)} \varphi d\omega - \int_T \varphi \widehat{P}_\varepsilon^*(d\omega) \right).$$

Thanks to the assumption (H₃), \widehat{P}_ε induces a bilipschitz equivalence of T onto its image. Therefore, by applying the area formula to the first integral in the right-hand side, we deduce

$$\begin{aligned} \langle \star d j(\mathbf{w}_\varepsilon) - \star d j(\widehat{\mathbf{v}}_\varepsilon), \varphi \rangle &= \sum_{T \in \mathcal{T}_\varepsilon} \int_T \left(\widehat{P}_\varepsilon^*(\varphi d\omega) - \varphi \widehat{P}_\varepsilon^*(d\omega) \right) \\ &= \sum_{T \in \mathcal{T}_\varepsilon} \int_T \left(\varphi \circ \widehat{P}_\varepsilon - \varphi \right) \widehat{P}_\varepsilon^*(d\omega) = \int_{\widehat{M}_\varepsilon} \left(\varphi \circ \widehat{P}_\varepsilon - \varphi \right) d j(\widehat{\mathbf{v}}_\varepsilon). \end{aligned}$$

The Hölder inequality and Lemma 3.8 then yield

$$\langle \star d j(\mathbf{w}_\varepsilon) - \star d j(\widehat{\mathbf{v}}_\varepsilon), \varphi \rangle \leq C \text{dist}(\widehat{M}_\varepsilon, M) \|\nabla \varphi\|_{L^\infty(\widehat{M}_\varepsilon)} \left(\|\widehat{\mathbf{v}}_\varepsilon\|_{L^2(\widehat{M}_\varepsilon)}^2 + \|\nabla \widehat{\mathbf{v}}_\varepsilon\|_{L^2(\widehat{M}_\varepsilon)}^2 \right)$$

whence, by applying (H₁) and (3.30), we conclude

$$(3.33) \quad \|\star d j(\widehat{\mathbf{v}}_\varepsilon) - \star d j(\mathbf{w}_\varepsilon)\|_{\text{flat}} \leq C\varepsilon |\log \varepsilon|.$$

Analysis of A_4 . Reminding the definition (3.23) of $\tilde{\mathbf{u}}_\varepsilon$ and Lemma 3.3, we have

$$(3.34) \quad \|\mathbf{w}_\varepsilon - \tilde{\mathbf{u}}_\varepsilon\|_{L^\infty(M)} = \|(\mathbf{w}_\varepsilon, \gamma)\|_{L^\infty(M)} \leq C\varepsilon.$$

On the other hand, using (3.30) and the assumption (H₃), we compute that

$$(3.35) \quad \|\nabla \mathbf{w}_\varepsilon\|_{L^2(M)} + \|\nabla \tilde{\mathbf{u}}_\varepsilon\|_{L^2(M)} \leq C |\log \varepsilon|^{1/2}.$$

We combine (3.27) in Lemma 3.7 with (3.34) and (3.35) to obtain

$$(3.36) \quad \|\star d j(\mathbf{w}_\varepsilon) - \star d j(\tilde{\mathbf{u}}_\varepsilon)\|_{\text{flat}} \leq C\varepsilon |\log \varepsilon|^{1/2}.$$

Analysis of A_5 . From the definition (3.23) of \mathbf{u}_ε and (3.35), we compute

$$(3.37) \quad \|\nabla \mathbf{u}_\varepsilon\|_{L^2(M)} \leq 2t_\varepsilon^{-1} \|\nabla \tilde{\mathbf{u}}_\varepsilon\|_{L^2(M)} \leq C t_\varepsilon^{-1} |\log \varepsilon|^{1/2}.$$

On the other hand, thanks to (3.34) we obtain

$$|\mathbf{u}_\varepsilon - \tilde{\mathbf{u}}_\varepsilon|^2 \leq (1 - |\tilde{\mathbf{u}}_\varepsilon|)^2 \leq (1 - |\mathbf{w}_\varepsilon|)^2 + C\varepsilon \leq (1 - |\mathbf{w}_\varepsilon|^2)^2 + C\varepsilon.$$

By integrating both sides of the inequality, and making a change of variable we deduce

$$(3.38) \quad \|\mathbf{u}_\varepsilon - \tilde{\mathbf{u}}_\varepsilon\|_{L^2(M)}^2 \leq \|1 - |\widehat{\mathbf{v}}_\varepsilon|^2\|_{L^2(\widehat{M}_\varepsilon)}^2 + C\varepsilon \mathcal{H}^2(\widehat{M}_\varepsilon) \leq C\varepsilon^2 |\log \varepsilon|.$$

For the last inequality, we have used (3.30) and the fact that $\mathcal{H}^2(\widehat{M}_\varepsilon) \leq C$, which follows from (H₃). Thus, by applying (3.27) (Lemma 3.7), with the help of (3.35), (3.37) and (3.38) we deduce that

$$(3.39) \quad \|\star d j(\tilde{\mathbf{u}}_\varepsilon) - \star d j(\mathbf{u}_\varepsilon)\|_{\text{flat}} \leq C t_\varepsilon^{-1} \varepsilon |\log \varepsilon|.$$

Analysis of A_6 . Arguing as in the proof of (3.31), and using (3.26) in Lemma 3.7 instead of Lemma 3.8, we obtain

$$(3.40) \quad \|\star \mathrm{d}j(\mathbf{u}_\varepsilon) - \mu_\varepsilon(\mathbf{v}_\varepsilon)\|_{\text{flat}} \leq \varepsilon |\log \varepsilon|.$$

Now, the proposition follows by combining (3.31), (3.32), (3.33), (3.36), (3.39) and (3.40). \square

4. THE ZERO-ORDER Γ -CONVERGENCE: EMERGENCE OF DEFECTS

4.1. Localized lower bounds for the energy. Thanks to Proposition 3.6, the compactness of the sequence $\widehat{\mu}_\varepsilon(\mathbf{v}_\varepsilon)$ is equivalent to the compactness of $\mu_\varepsilon(\mathbf{v}_\varepsilon)$. The latter is defined in terms of the fields $\mathbf{w}_\varepsilon \in W^{1,\infty}(M; \mathbb{R}^3)$, given by (3.23), which interpolate continuously the discrete fields \mathbf{v}_ε but are not necessarily tangent nor unit-valued. We discuss now a localized lower bound for the Dirichlet energy of \mathbf{w}_ε . Similar results are well-known in the continuum Ginzburg-Landau setting, where they play a major rôle (see, e.g., [15, Theorems 2.1 and 4.1] and [26, Theorem 1]), and are also available for the discrete XY-energy [2, Proposition 3.2]. Given a point $x_0 \in M$ and a radius $\rho > 0$, we denote by $B_\rho(x_0)$ the geodesic ball of centre x_0 and radius ρ . We follow the approach in [15]. We define

$$(4.1) \quad \alpha_\varepsilon := (1 - C\varepsilon)^{-2} \operatorname{ess\,inf}_{x \in M} \inf_{\mathbf{X} \in T_x M, |\mathbf{X}|=1} (\mathbf{A}_\varepsilon^{-1}(x)\mathbf{X}, \mathbf{X}) (\det \mathbf{A}_\varepsilon(x))^{1/2}$$

and, for $0 < \varepsilon < \rho$ and $d \in \mathbb{Z}$, we let

$$(4.2) \quad \lambda_\varepsilon(\rho, d) := \alpha_\varepsilon \min_{0 \leq m \leq 1} \left\{ \frac{\pi|d|}{\rho + C\rho^2} m^2 \vee \frac{C}{\varepsilon} (1 - m)^3 \right\}.$$

The constant C , which will be selected below, does not depend on ε , ρ and d . Recall that, given a vector field $\mathbf{v}_\varepsilon \in T(\mathcal{T}_\varepsilon; \mathbb{S}^2)$, we let $\mathbf{w}_\varepsilon := \widehat{\mathbf{v}}_\varepsilon \circ \widehat{P}_\varepsilon^{-1}$ and we denote by \mathbf{u}_ε the vector field defined by (3.23).

Lemma 4.1. *There exist positive constants R_* and ε_* with the following property: for any $\varepsilon \in (0, \varepsilon_*]$, any $x_0 \in M$, any $\rho \in (\varepsilon, R_*]$ and any $\mathbf{v}_\varepsilon \in T_\varepsilon(\mathcal{T}_\varepsilon; \mathbb{S}^2)$ such that $|\mathbf{w}_\varepsilon| \geq 1/2$ on $\partial B_\rho(x_0)$, there holds*

$$(4.3) \quad \frac{1}{2} \|\mathbf{w}_\varepsilon\|_{W_\varepsilon^{1,2}(\partial B_\rho(x_0))}^2 \geq \lambda_\varepsilon(\rho, \operatorname{ind}(\mathbf{u}_\varepsilon, \partial B_\rho(x_0))).$$

Moreover, there holds

$$(4.4) \quad \lambda_\varepsilon(\rho, d) \geq \frac{(1 - C\varepsilon)\pi|d|}{\rho + C\rho^2} - C\varepsilon^{1/3}|d|^{1/3}\rho^{-4/3}$$

for any $\rho > \varepsilon$, $d \in \mathbb{Z}$ and some constant C which does not depend on ε , ρ , d .

Proof of Lemma 4.1. This argument is adapted from [15, Theorem 2.1]. Throughout the proof, the symbol C denotes several constants that do not depend on ε or $x_0 \in M$, but possibly on R_* , ε_* . We start by proving (4.3).

Step 1. There exist positive numbers R_* and C such that, for any $x_0 \in M$ and any $0 < \rho \leq R_*$, there holds

$$(4.5) \quad \mathcal{H}^1(\partial B_\rho(x_0)) \leq 2\pi\rho + C\rho^2.$$

Indeed, thanks to the area formula, the left-hand side is bounded by $\operatorname{Lip}(\exp_{x_0}|_{D_\rho}) \mathcal{H}^1(\partial D_\rho)$, where $\exp_{x_0}: T_{x_0}M \rightarrow M$ is the exponential map and $D_\rho \subseteq T_{x_0}M$ is the disk of radius ρ centred at the origin. Since $D(\exp_{x_0}) = \operatorname{Id}_{T_{x_0}M}$, by a Taylor expansion we see that $\operatorname{Lip}(\exp_{x_0}|_{D_\rho}) \leq 1 + C\rho$ if ρ is small enough. By a compactness argument, the numbers R_* and C can be chosen uniformly with respect to x_0 .

Step 2. Take a point $x_0 \in M$ and a field $\mathbf{v}_\varepsilon \in T(\mathcal{T}_\varepsilon; \mathbb{S}^2)$ such that $|\mathbf{w}_\varepsilon| \geq 1/2$ on $\partial B_\rho(x_0)$. Since x_0 will be fixed for the rest of the proof, we will omit it from the notation. Thanks to Lemma 3.3 and to (3.22), (3.23) we have that $|\tilde{\mathbf{u}}_\varepsilon| \geq 1/4$ and $\mathbf{u}_\varepsilon = \tilde{\mathbf{u}}_\varepsilon/|\tilde{\mathbf{u}}_\varepsilon|$ on ∂B_ρ provided that ε is small enough, say, smaller than some number ε_* . Then, we compute

$$\begin{aligned} |\mathbf{w}_\varepsilon| |(\gamma, \mathbf{u}_\varepsilon \times \nabla_\tau \mathbf{u}_\varepsilon)| &= \frac{|\mathbf{w}_\varepsilon|}{|\tilde{\mathbf{u}}_\varepsilon|^2} |(\gamma, \tilde{\mathbf{u}}_\varepsilon \times \nabla_\tau \tilde{\mathbf{u}}_\varepsilon)| \\ &= \frac{|\mathbf{w}_\varepsilon|}{|\tilde{\mathbf{u}}_\varepsilon|^2} |(\gamma, \mathbf{w}_\varepsilon \times \nabla_\tau \mathbf{w}_\varepsilon) - (\mathbf{w}_\varepsilon, \gamma) (\gamma, \mathbf{w}_\varepsilon \times \nabla_\tau \gamma)| \\ &\leq \frac{|\mathbf{w}_\varepsilon|^2}{|\tilde{\mathbf{u}}_\varepsilon|^2} (|\nabla \mathbf{w}_\varepsilon| + C|(\mathbf{w}_\varepsilon, \gamma)|). \end{aligned}$$

Using (3.23) and the fact that $|\tilde{\mathbf{u}}_\varepsilon| \geq 1/4$, we can bound the ratio $|\mathbf{w}_\varepsilon|/|\tilde{\mathbf{u}}_\varepsilon|$ in terms of $|(\mathbf{w}_\varepsilon, \gamma)|$, which in turns is bounded by $C\varepsilon$, due to Lemma 3.3. This yields

$$|\mathbf{w}_\varepsilon| |(\gamma, \mathbf{u}_\varepsilon \times \nabla_\tau \mathbf{u}_\varepsilon)| \leq (1 + C\varepsilon) (|\nabla \mathbf{w}_\varepsilon| + C\varepsilon).$$

After a rearrangement, and using again that $|\mathbf{w}_\varepsilon| \geq 1/2$, we obtain

$$(4.6) \quad |\nabla \mathbf{w}_\varepsilon|^2 \geq (1 + C\varepsilon)^{-2} |\mathbf{w}_\varepsilon|^2 (|(\gamma, \mathbf{u}_\varepsilon \times \nabla_\tau \mathbf{u}_\varepsilon)| - C\varepsilon)^2.$$

Step 3. Thanks to Lemma 3.1, there exists ε_* such that the quantity α_ε defined in (4.1) satisfies

$$(4.7) \quad \alpha_\varepsilon \geq 1 - C\varepsilon > 0 \quad \text{for any } 0 < \varepsilon \leq \varepsilon_*.$$

Fix ε and ρ such that $0 < \varepsilon \leq \varepsilon_*$ and $\varepsilon < \rho \leq R_*$. By definition (3.3) of the $W_\varepsilon^{1,2}$ -seminorm, there holds

$$(4.8) \quad \frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(\partial B_\rho)}^2 \geq \frac{\alpha_\varepsilon (1 - C\varepsilon)^2}{2} \int_{\partial B_\rho} |\nabla \mathbf{w}_\varepsilon|^2 \, ds.$$

Set $m(\rho) := \min_{\partial B_\rho} |\mathbf{w}_\varepsilon|$ (note that $m(\rho) \in [1, 1/2]$), and let τ be a unit tangent field on ∂B_ρ . Using (4.6) and the definition (3.9) of j , we obtain

$$\frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(\partial B_\rho)}^2 \geq \frac{\alpha_\varepsilon m^2(\rho)}{2} \int_{\partial B_\rho} |j(\mathbf{u}_\varepsilon), \tau - C\varepsilon|^2 \, ds.$$

By applying Jensen inequality, we deduce

$$\begin{aligned} \frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(\partial B_\rho)}^2 &\geq \frac{\alpha_\varepsilon m^2(\rho)}{2\mathcal{H}^1(\partial B_\rho)} \left| \int_{\partial B_\rho} j(\mathbf{u}_\varepsilon) - C\varepsilon \mathcal{H}^1(\partial B_\rho) \right|^2 \\ &\stackrel{(4.5)}{\geq} \frac{\alpha_\varepsilon m^2(\rho)}{4\pi\rho + C\rho^2} \left| \int_{\partial B_\rho} j(\mathbf{u}_\varepsilon) - C\varepsilon \mathcal{H}^1(\partial B_\rho) \right|^2. \end{aligned}$$

Using Lemma 3.5, and arguing as in the proof of (3.25), we can evaluate the integral of $j(\mathbf{u}_\varepsilon)$ in terms of $d := \text{ind}(\mathbf{u}_\varepsilon, \partial B_\rho)$ and the Gauss curvature G :

$$\frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(\partial B_\rho)}^2 \geq \frac{\alpha_\varepsilon m^2(\rho)}{4\pi\rho + C\rho^2} \left| 2\pi d - \int_{B_\rho} G - C\varepsilon \mathcal{H}^1(\partial B_\rho) \right|^2.$$

Now, the Gauss curvature G is bounded and $\varepsilon < \rho \leq R_*$, so the integral of G is uniformly bounded by CR_*^2 , while $\varepsilon \mathcal{H}^1(\partial B_\rho) \leq CR_*^2$. Thus, reducing the value of R_* if necessary, we can assume that

$$u := \frac{1}{2\pi} \left| \int_{B_\rho} G + C\varepsilon \mathcal{H}^1(\partial B_\rho) \right| < \frac{1}{2}.$$

If $|d| > 1$, then $|d - u|^2 \geq d^2 - 2u|d| \geq |d|$ and hence

$$(4.9) \quad \frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(\partial B_\rho)}^2 \geq \frac{\alpha_\varepsilon \pi m^2(\rho)}{\rho + C\rho^2} |d|.$$

An elementary computation, based on the fact that $u \leq C\rho^2$, shows that the same inequality is satisfied also if $|d| = 1$, provided that we take a larger constant C in the denominator, and we reduce again the value of R_* if necessary. Finally, (4.9) is trivially satisfied if $d = 0$.

Step 4. Suppose that $m(\rho) < 1$, and let $x_\rho \in \partial B_\rho$ be such that $|\mathbf{w}_\varepsilon(x_\rho)| = m(\rho)$. We have $|\mathbf{w}_\varepsilon| \leq (1 + m(\rho))/2$ on $B_{\rho'}(x_\rho)$, where

$$\rho' := \frac{1 - m(\rho)}{2 \text{Lip}(\mathbf{w}_\varepsilon)}.$$

Now, \mathbf{w}_ε has Lipschitz constant $\text{Lip}(\mathbf{w}_\varepsilon) \leq C\varepsilon^{-1}$, thanks to Lemma 3.2. Thus, since we have assumed that $\rho > \varepsilon$, we conclude that

$$\mathcal{H}^1(\partial B_\rho \cap B_{\rho'}(x_\rho)) \geq C\rho' \geq C\varepsilon(1 - m(\rho)).$$

Thus, by applying Lemma 3.4, we estimate

$$(4.10) \quad \frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(\partial B_\rho)}^2 \geq \frac{C}{\varepsilon^2} \int_{\partial B_\rho \cap B_{\rho'}(x_\rho)} (1 - |\mathbf{w}_\varepsilon|^2)^2 \, ds \geq \frac{C}{\varepsilon} (1 - m(\rho))^3.$$

Note that (4.10) trivially holds when $m(\rho) = 1$. Now, (4.3) follows by combining (4.7), (4.9) and (4.10).

Step 5 (Proof of (4.4)). The function $f: m \in [0, 1] \mapsto Am^2 \vee B(1 - m)^3$ achieves its minimum value at the point m_0 such that $Am_0^2 = B(1 - m_0)^3$. From this equality, we deduce that

$$\frac{B}{A} (1 - m_0)^3 = m_0^2 \leq 1,$$

whence $m_0 \geq 1 - (A/B)^{1/3}$ and $f(m_0) \geq A(1 - 2(A/B)^{1/3})$. Substituting for A and B the expressions in (4.2), and using (4.1), yields (4.4). \square

Following Jerrard [15], it will be useful to reformulate the lower bound (4.3) in terms of a function Λ_ε , defined by

$$(4.11) \quad \Lambda_\varepsilon(r) := \int_0^r \lambda_\varepsilon(\rho, 1) \wedge \frac{C_*}{\varepsilon} \, d\rho \quad \text{for } r > 0.$$

We first collect a few properties of Λ_ε (see also [15, Proposition 3.1]).

Lemma 4.2. *The function Λ_ε satisfies*

$$(4.12) \quad \Lambda_\varepsilon(r + s) \leq \Lambda_\varepsilon(r) + \Lambda_\varepsilon(s), \quad \Lambda_\varepsilon(r) \leq \Lambda_\varepsilon(s), \quad \frac{\Lambda_\varepsilon(r)}{r} \geq \frac{\Lambda_\varepsilon(s)}{s}$$

for any $0 < r \leq s$. Moreover, there holds

$$(4.13) \quad \Lambda_\varepsilon(r) \geq (1 - C\varepsilon)\pi \log \frac{r}{\varepsilon} - C$$

for any $r \in (\varepsilon, R_*)$ (where R_* is given by Lemma 4.1) and some ε -independent constant C .

Proof. It is clear by the definition (4.2) that λ_ε is positive and decreasing; then (4.12) follows by elementary calculus. As for the lower bound (4.13), Equation (4.4) implies that

$$\lambda_\varepsilon(\rho, 1) \wedge \frac{C}{\varepsilon} \geq \frac{(1 - C\varepsilon)\pi}{\rho + C\rho^2} - C\varepsilon^{1/3}\rho^{-4/3}$$

for any $\rho \in (c_1\varepsilon, R_*]$, for some constant $c_1 > 0$. By integrating both sides of this inequality with respect to $\rho \in (c_1\varepsilon, r)$, we deduce

$$\begin{aligned} \Lambda_\varepsilon(r) &\geq (1 - C\varepsilon)\pi \left\{ \log \frac{r}{c_1\varepsilon} + \log \frac{C\varepsilon + 1}{Cr + 1} \right\} + C\varepsilon^{1/3} (R^{1/3} - r^{1/3}) \\ &\stackrel{c_1\varepsilon < r < R_*}{\geq} (1 - C\varepsilon)\pi \log \frac{r}{\varepsilon} - C, \end{aligned}$$

where the constant C in the right-hand side depends only on c_1 and R_* . If $c_1 \leq 1$, then the lemma follows immediately. Otherwise, we note that, by choosing C large enough, the right-hand side of (4.13) can be made non-positive for every $r \in [\varepsilon, c_1\varepsilon]$, so that (4.13) holds trivially. \square

We state a lower bound for the energy on annuli in terms of the function Λ_ε .

Lemma 4.3. *For any $\varepsilon \in (0, \varepsilon_*]$, any $x_0 \in M$, any $\varepsilon < r < R \leq R_*$ (where ε_* , R_* are given by Lemma 4.1) and any $\mathbf{v}_\varepsilon \in \mathcal{T}_\varepsilon(\mathcal{T}_\varepsilon; \mathbb{S}^2)$ such that $|\mathbf{w}_\varepsilon| \geq 1/2$ on $A_{r,R} := B_R(x_0) \setminus B_r(x_0)$, there holds*

$$\frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(A_{r,R})}^2 \geq |d| \left\{ \Lambda_\varepsilon \left(\frac{R}{|d|} \right) - \Lambda_\varepsilon \left(\frac{r}{|d|} \right) \right\},$$

where $d := \text{ind}(\mathbf{u}_\varepsilon, \partial B_R(x_0))$.

The proof of this lemma follows by integrating the lower bound (4.3); see [15, Proposition 3.2] for details.

4.2. The ball construction. In this section, we recall the “ball construction” as presented by Jerrard [15] (a similar construction was independently introduced by Sandier [26]). In contrast with [15], our lower bound (Lemma 4.3) is only valid for annuli with outer radius $\leq R_*$, so we need to make sure that this constraint is preserved by the construction.

Throughout this section, we fix a sequence of discrete fields $\mathbf{v}_\varepsilon \in \mathcal{T}(\mathcal{T}_\varepsilon; \mathbb{S}^2)$, for $\varepsilon \in (0, \varepsilon_*]$, that satisfies the logarithmic energy bound (H). We define the set $S_\varepsilon := \{x \in M : |\mathbf{w}_\varepsilon(x)| \leq 1/2\}$ and the measure

$$(4.14) \quad \nu_\varepsilon(\mathbf{v}_\varepsilon) := \frac{1}{2\pi} (\mu_\varepsilon(\mathbf{v}_\varepsilon) + G \, dS),$$

where $\mu_\varepsilon(\mathbf{v}_\varepsilon)$ is given by (3.24). Since (\mathbf{v}_ε) is fixed, throughout this section we write ν_ε instead of $\nu_\varepsilon(\mathbf{v}_\varepsilon)$. If $E \subseteq M$ is a Borel set with $\partial E \subseteq M \setminus S_\varepsilon$, Equation (3.25) implies that $\nu_\varepsilon(E) = \text{ind}(\mathbf{u}_\varepsilon, \partial E)$. The sequence (ν_ε) is precompact in the flat topology if and only if $(\mu_\varepsilon(\mathbf{v}_\varepsilon))$ is, and hence (by Proposition 3.6), if and only if $(\hat{\mu}_\varepsilon(\mathbf{v}_\varepsilon))$ is. We will also need the following notation: given a closed ball B , we denote by $\text{rad}(B)$ its radius. If \mathcal{B} is a finite collection of closed balls, we set $\text{spt } \mathcal{B} := \cup_{B \in \mathcal{B}} B$.

Lemma 4.4. *There exists an ε -independent constant β such that, for any $T \in \mathcal{T}_\varepsilon$,*

$$P(T) \cap S_\varepsilon \neq \emptyset \quad \text{implies} \quad \frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(P(T))}^2 \geq \beta.$$

Proof. Suppose that there is a point $x_0 \in P(T)$ such that $|\mathbf{w}_\varepsilon(x_0)| \leq 1/2$. Arguing as in the proof of Lemma 4.1, Step 4, we deduce that $|\mathbf{w}_\varepsilon| \leq 3/4$ on a ball $B_{\rho'}(x_0)$ with $\rho' \geq C\varepsilon$. Then, using also the assumptions (H₁) and (H₃), we see that $\mathcal{H}^2(P(T) \cap B_{\rho'}(x_0)) \geq C\varepsilon$ and hence, by Lemma 3.4, we estimate

$$\frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(P(T))}^2 \geq \frac{C}{\varepsilon^2} \int_{P(T) \cap B_{\rho'}(x_0)} (1 - |\mathbf{w}_\varepsilon|^2)^2 \, dS \geq C. \quad \square$$

Lemma 4.5. *There exists an ε -independent constant C such that, for any $T \in \mathcal{T}_\varepsilon$, there holds $|\nu_\varepsilon(P(T))| \leq C$.*

Proof. We have that

$$|\nu_\varepsilon(P(T))| \stackrel{(4.14)}{\leq} |\mu_\varepsilon(\mathbf{v}_\varepsilon)[P(T)]| + C\varepsilon^2 \stackrel{(3.24)}{=} \left| \int_{\partial(P(T))} j(\mathbf{u}_\varepsilon) \right| + C\varepsilon^2$$

(we have used that the Gauss curvature is bounded and the surface area of $P(T)$ is $\leq C\varepsilon^2$, which follows from (H_1) and (H_3)). Using now the definition (3.23) of \mathbf{u}_ε , we compute that $j(\mathbf{u}_\varepsilon) = \eta_\varepsilon(|\tilde{\mathbf{u}}_\varepsilon|)j(\tilde{\mathbf{u}}_\varepsilon)$ and hence $|j(\mathbf{u}_\varepsilon)| \leq \text{Lip}(\tilde{\mathbf{u}}_\varepsilon)$. Combining (3.23) with the Lipschitz bound (3.5) for \mathbf{w}_ε , we see that $\text{Lip}(\tilde{\mathbf{u}}_\varepsilon) \leq \text{Lip}(\mathbf{w}_\varepsilon) \leq C\varepsilon^{-1}$. Thus

$$|\nu_\varepsilon(P(T))| \leq C\varepsilon^{-1} \mathcal{H}^1(\partial P(T)) + C\varepsilon^2 \leq C,$$

where we have used that $\mathcal{H}^1(\partial P(T)) \leq C\varepsilon$, due to (H_1) and (H_3) . \square

For any $T \in \mathcal{T}_\varepsilon$ such that $P(T) \cap S_\varepsilon \neq \emptyset$, we consider the smallest closed ball B of centre $P(x_T)$ such that $P(T) \subseteq B$. Let \mathcal{B}_ε be the collection of such balls. Thanks to the assumption (H_1) , any $B \in \mathcal{B}_\varepsilon$ satisfies

$$(4.15) \quad C^{-1}\varepsilon \leq \text{rad}(B) \leq C\varepsilon.$$

Moreover, each ball $B \in \mathcal{B}_\varepsilon$ intersects $P(T)$ for at most C triangles $T \in \mathcal{T}_\varepsilon$, where C is an ε -independent constant. Therefore, from (4.15) and Lemma 4.5 we deduce that

$$(4.16) \quad C^{-1}\varepsilon \leq s_\varepsilon := \min_{B \in \mathcal{B}_\varepsilon} \frac{\text{rad}(B)}{|\nu_\varepsilon(B)|} \leq C\varepsilon.$$

(To prove the upper bound, note that $\text{spt } \mathcal{B}_\varepsilon \supseteq \text{spt}(\nu_\varepsilon)$ and that $|\nu_\varepsilon(B)| \geq 1$ as soon as $B \cap \text{spt}(\nu_\varepsilon) \neq \emptyset$. Here, we are assuming WLOG that $\nu_\varepsilon \not\equiv 0$, otherwise $s_\varepsilon = +\infty$). Finally, as a consequence of Lemma 4.4 and the energy bound (H), we obtain

$$(4.17) \quad \#(\mathcal{B}_\varepsilon) \leq C|\log \varepsilon|.$$

The following proposition is adapted from [15, Proposition 4.1] (see also [28, Proposition 5.4]).

Proposition 4.6. *There exists an $(\varepsilon$ -independent) positive constant C such that, for any $s \in [s_\varepsilon, CR_*\#(\mathcal{B}_\varepsilon)^{-1}]$, there exists a family of pairwise disjoint, closed balls $\mathcal{B}_\varepsilon(s)$ with the following properties.*

- (i) $\text{spt } \mathcal{B}_\varepsilon \subseteq \text{spt } \mathcal{B}_\varepsilon(s) \subseteq \text{spt } \mathcal{B}_\varepsilon(t)$ for any $s_\varepsilon \leq s \leq t \leq CR_*\#(\mathcal{B}_\varepsilon)^{-1}$.
- (ii) For any $B \in \mathcal{B}_\varepsilon(s)$, there holds

$$\frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(B \setminus \text{spt } \mathcal{B}_\varepsilon)}^2 \geq \frac{\text{rad}(B)}{s} (\Lambda_\varepsilon(s) - \Lambda_\varepsilon(s_\varepsilon)).$$

- (iii) For any $B \in \mathcal{B}_\varepsilon(s)$, there holds $\text{rad}(B) \geq s|\nu_\varepsilon(B)|$.
- (iv) There holds

$$\sum_{B \in \mathcal{B}_\varepsilon(s)} \text{rad}(B) \leq \frac{s}{s_\varepsilon} \sum_{B \in \mathcal{B}_\varepsilon} \text{rad}(B).$$

Sketch of the proof. If the balls in \mathcal{B}_ε are not pairwise disjoint, the construction starts with a *merging* phase, that is, we select a pair of balls $B, B' \in \mathcal{B}_\varepsilon$ such that $B \cap B' \neq \emptyset$ and we replace them with a new ball B^{new} such that $B^{\text{new}} \supseteq B \cup B'$, $\text{rad}(B^{\text{new}}) = \text{rad}(B) + \text{rad}(B')$. We repeat this operation until we obtain a collection of pairwise disjoint balls, which we call \mathcal{B}'_ε . If all the balls in the original collection \mathcal{B}_ε were pairwise disjoint, then $\mathcal{B}_\varepsilon = \mathcal{B}'_\varepsilon$. Set $\mathcal{B}_\varepsilon(s_\varepsilon) := \mathcal{B}_\varepsilon$, so (i), (iii), (iv) are trivially satisfied and (ii) is also satisfied, because of (4.16).

Now we perform an *expansion* phase, i.e. we let the parameter s grow continuously, and we let the “minimizing balls” (i.e., the balls B such that $\text{rad}(B) = s|\nu_\varepsilon(B)|$) grow, leaving the other

unchanged. More precisely, if $\mathcal{B}'_\varepsilon = \{B_i\}_{i=1}^k$ and x_i is the centre of B_i , then the elements of $\mathcal{B}_\varepsilon(s)$ are defined by

$$(4.18) \quad B_i(s) := \begin{cases} B_i & \text{if } \text{rad}(B_i) > s|\nu_\varepsilon(B_i)| \\ B_i(x_i, s|\nu_\varepsilon(B_i)|) & \text{otherwise.} \end{cases}$$

For s small enough, the balls $B_i(s)$'s are pairwise disjoint. We also have $|\nu_\varepsilon(B_i)| = |\nu_\varepsilon(B_i(s))|$, because $(B_i(s) \setminus B_i) \cap \text{spt } \mathcal{B}_\varepsilon = \emptyset$ and $\text{spt}(\nu_\varepsilon) \subseteq \text{spt } \mathcal{B}_\varepsilon$. If for some s^* there happens $B_i(s^*) \cap B_j(s^*) \neq \emptyset$ for $i \neq j$, then we stop the expansion phase. We define $\mathcal{B}_\varepsilon(s^*)$ as the family of balls obtained from $\{B_i(s^*)\}_{i=1}^k$ via merging. For $s > s^*$, we repeat an expansion phase according to the same rule as (4.18), until two or more balls touch and we perform a merging phase again, and so on.

Arguing as in [15, Proposition 4.1], one can show that $\mathcal{B}_\varepsilon(s)$ satisfies (i), (ii) and (iii). (Actually, (ii) appears in a slightly different form, but the same argument applies.) The proof of (ii) relies on Lemma 4.3; in order to apply this lemma, we need to make sure that the radii of all the balls we consider are $\leq R_*$. However, if we temporarily assume that (iv) holds, then (using (4.15) and (4.16) as well) we see that

$$\sum_{B \in \mathcal{B}_\varepsilon(s)} \text{rad}(B) \leq Cs \#(\mathcal{B}_\varepsilon).$$

Therefore, we have $\text{rad}(B) \leq R_*$ for any $B \in \mathcal{B}_\varepsilon(s)$ and any $s \leq C^{-1}R_*\#(\mathcal{B}_\varepsilon)$.

To prove (iv), we note that the quantity $\sum_{B \in \mathcal{B}_\varepsilon(s)} \text{rad}(B)$ is preserved during each merging phase. Then, for a fixed s , let $s_1 < \dots < s_k < s$ be the values of the parameter when merging occurred, and take $B_i(s) \in \mathcal{B}_\varepsilon(s)$. From (4.18), we see that

$$\text{rad}(B_i(s)) = \min \{ \text{rad}(B_i(s_k)), s|\nu_\varepsilon(B_i(s))| \} \leq s|\nu_\varepsilon(B_i(s_k))| \stackrel{(iii)}{\leq} \frac{s}{s_k} \text{rad}(B_i(s_k))$$

(we have used that $s \mapsto |\nu_\varepsilon(B_i(s))|$ is constant during each expansion phase). Thus,

$$\sum_{B \in \mathcal{B}_\varepsilon(s)} \text{rad}(B) \leq \frac{s}{s_k} \sum_{B \in \mathcal{B}_\varepsilon(s_k)} \text{rad}(B).$$

Now, we complete the proof of (iv) arguing by induction. \square

As an immediate consequence of Proposition 4.6, using (4.15), (4.16) the definition (4.11) of Λ_ε and (4.13) in Lemma 4.2, we obtain

Corollary 4.7. *For any $s \in [s_\varepsilon, CR_*\#(\mathcal{B}_\varepsilon)^{-1}]$, there exists a family of pairwise disjoint, closed balls $\mathcal{B}_\varepsilon(s)$ which satisfies the following properties:*

- (i) $\text{spt } \mathcal{B}_\varepsilon \subseteq \text{spt } \mathcal{B}_\varepsilon(s) \subseteq \text{spt } \mathcal{B}_\varepsilon(t)$ for any $s_\varepsilon \leq s \leq t \leq R_*\#(\mathcal{B}_\varepsilon)^{-1}$;
- (ii) for any $B \in \mathcal{B}_\varepsilon(s)$, there holds

$$\frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(B \setminus \text{spt } \mathcal{B}_\varepsilon)}^2 \geq |\nu_\varepsilon(B)| \left(\pi(1 - C\varepsilon) \log \frac{s}{\varepsilon} - C \right);$$

- (iii) there holds $\sum_{B \in \mathcal{B}_\varepsilon(s)} \text{rad}(B) \leq Cs \#(\mathcal{B}_\varepsilon)$.

4.3. Proof of the zero-order Γ -convergence. We state and prove a zero-order Γ -convergence result in terms of the measures $\nu_\varepsilon(\mathbf{v}_\varepsilon)$. Given a measure $\mu \in X$ with $\mu = \sum_i d_i \delta_{x_i}$, we set

$$\sigma_0(\mu) := \frac{1}{2} \min \left\{ \min_{j \neq i} \text{dist}(x_i, x_j), \text{injectivity radius of } M \right\}.$$

Proposition 4.8. *Suppose that the assumptions (H_1) , (H_2) and (H_3) are satisfied. Then, the following results hold.*

- (i) *Compactness. If (\mathbf{v}_ε) is a sequence in $T(\mathcal{T}_\varepsilon; \mathbb{S}^2)$ that satisfies the energy bound (H) then, up to subsequences, $\nu_\varepsilon(\mathbf{v}_\varepsilon) \xrightarrow{\text{flat}} \mu$ for some $\mu \in X$.*
- (ii) *Localized Γ -liminf inequality. Let (\mathbf{v}_ε) be a sequence in $T(\mathcal{T}_\varepsilon; \mathbb{S}^2)$ such that $\nu_\varepsilon(\mathbf{v}_\varepsilon) \xrightarrow{\text{flat}} \mu$ for some $\mu \in X$, $\mu = \sum_{i=1}^K d_i \delta_{x_i}$. Then, there exists a constant C such that, for any $i \in \{1, \dots, K\}$ and any $0 < \sigma \leq \sigma_0(\mu)$, there holds*

$$\liminf_{\varepsilon \rightarrow 0} \left(\frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(B_\sigma(x_i))}^2 - \pi |d_i| \log \frac{\sigma}{\varepsilon} \right) \geq C.$$

- (iii) *Γ -limsup inequality. For any $\mu \in X$ there exists a sequence (\mathbf{v}_ε) in $T(\mathcal{T}_\varepsilon; \mathbb{S}^2)$ such that $\nu_\varepsilon(\mathbf{v}_\varepsilon) \xrightarrow{\text{flat}} \mu$ and*

$$\limsup_{\varepsilon \rightarrow 0} \frac{|\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(M)}^2}{2|\log \varepsilon|} \leq \pi |\mu|(M).$$

The proof of this Proposition is adapted from [2, Theorem 3.1]. Throughout the proof, we write ν_ε instead of $\nu_\varepsilon(\mathbf{v}_\varepsilon)$ when no confusion is possible.

Proof of (i) — Compactness. Let $\mathcal{B}_\varepsilon^1 := \mathcal{B}_\varepsilon(s_\varepsilon^1)$ be the family of balls given by Corollary 4.7 for the choice of parameter $s_\varepsilon^1 := \varepsilon^{1/2}$. If ε is small enough, we have

$$s_\varepsilon \stackrel{(4.16)}{\leq} \varepsilon^{1/2} \leq \frac{CR_*}{|\log \varepsilon|} \stackrel{(4.17)}{\leq} \frac{CR_*}{\#(\mathcal{B}_\varepsilon)},$$

so s_ε^1 satisfies the assumptions of Corollary 4.7. By (i) in Corollary 4.7, we have that $\text{spt}(\nu_\varepsilon) \subseteq \text{spt} \mathcal{B}_\varepsilon \subseteq \text{spt} \mathcal{B}_\varepsilon^1$, while (ii) implies

$$\frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(B)}^2 \geq |\nu_\varepsilon(B)| \left(\frac{\pi}{2} (1 - C\varepsilon) |\log \varepsilon| - C \right)$$

for any $B \in \mathcal{B}_\varepsilon^1$. Summing up this inequality over all the B 's, using Lemma 3.2 and the energy bound (H), one sees that

$$\sum_{B \in \mathcal{B}_\varepsilon^1} |\nu_\varepsilon(B)| \leq C$$

for an ε -independent constant C . Therefore, the measures $\nu_\varepsilon^1 := \sum_{B \in \mathcal{B}_\varepsilon^1} \nu_\varepsilon(B) \delta_{x(B)} \in X$, where $x(B)$ denotes the centre of the ball B , have uniformly bounded mass and flat-converge to an element of X , up to extraction of a subsequence. On the other hand, (iii) in Corollary 4.7 implies

$$\sum_{B \in \mathcal{B}_\varepsilon^1} \text{rad}(B) \leq C \varepsilon^{1/2} \#(\mathcal{B}_\varepsilon) \stackrel{(4.17)}{\leq} C \varepsilon^{1/2} |\log \varepsilon|.$$

Then, arguing as in [2, Theorem 3.1.(i)], one can show that $\|\nu_\varepsilon - \nu_\varepsilon^1\|_{\text{flat}} \rightarrow 0$, which yields compactness for the sequence (ν_ε) . (see also [3, Theorem 3.3] for more details). \square

Proof of (ii) — Γ -liminf. Fix $i \in \{1, \dots, K\}$ and $0 < \sigma \leq \sigma_0(\mu)$. By extraction of a non-relabelled subsequence, we can assume WLOG that

$$(4.19) \quad \lim_{\varepsilon \rightarrow 0} \left(\frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(B_\sigma(x_i))}^2 - \pi |d_i| |\log \varepsilon| \right) < +\infty$$

(and, in particular, the limit exists). Arguing as in [2, Theorem 3.1.(ii)], we see that $\|\nu_\varepsilon(\mathbf{v}_\varepsilon) - \nu_\varepsilon(\bar{\mathbf{v}}_\varepsilon)\|_{\text{flat}} \rightarrow 0$, where $\bar{\mathbf{v}}_\varepsilon$ denotes the restriction of \mathbf{v}_ε to $B_\sigma(x_i)$, and that $\nu_\varepsilon(\bar{\mathbf{v}}_\varepsilon)$ flat-converges

to $d_i \delta_{x_i}$. Therefore, we can repeat the ball construction of Section 4.2 with M replaced by $B_\sigma(x_i)$ and \mathbf{v}_ε replaced by $\bar{\mathbf{v}}_\varepsilon$. (This guarantees that no ball “coming from outside” enters $B_\sigma(x_i)$.) We still write ν_ε instead of $\nu_\varepsilon(\bar{\mathbf{v}}_\varepsilon)$.

For a fixed $\gamma \in (0, 1)$, we apply Corollary 4.7 with $s = s_\varepsilon^2 := \varepsilon^\gamma$. (One can check, arguing as in the proof of (i), that the assumptions of Corollary 4.7 are satisfied.) The collection of balls $\mathcal{B}_\varepsilon(s_\varepsilon^2)$ satisfies $\text{spt}(\nu_\varepsilon) \subseteq \text{spt } \mathcal{B}_\varepsilon(s_\varepsilon^2)$,

$$\sum_{B \in \mathcal{B}_\varepsilon(s_\varepsilon^2)} \text{rad}(B) \leq C\varepsilon^\gamma \#(\mathcal{B}_\varepsilon) \stackrel{(4.17)}{\leq} C\varepsilon^\gamma |\log \varepsilon|$$

and

$$(4.20) \quad \frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(B)}^2 \geq |\nu_\varepsilon(B)| (\pi(1-\gamma)(1-C\varepsilon) |\log \varepsilon| - C)$$

for any $B \in \mathcal{B}_\varepsilon(s_\varepsilon^2) \setminus \mathcal{B}_\varepsilon$. Let $\mathcal{B}_\varepsilon^2 := \{B \in \mathcal{B}_\varepsilon(s_\varepsilon^2) : B \subseteq B_\sigma(x_i)\}$ and $\nu_\varepsilon^2 := \sum_{B \in \mathcal{B}_\varepsilon^2} \nu_\varepsilon(B) \delta_{x(B)}$. Arguing again as in [2], we see that $\|\nu_\varepsilon^2 - \nu_\varepsilon\|_{\text{flat}} \rightarrow 0$, so ν_ε^2 flat-converges to $d_i \delta_{x_i}$ and, in particular, $\liminf_{\varepsilon \rightarrow 0} |\nu_\varepsilon^2|(B_\sigma(x_i)) \geq |d_i|$. Now, by summing up the inequality (4.20) with respect to $B \in \mathcal{B}_\varepsilon^2$, we deduce

$$\frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(B_\sigma(x_i))}^2 \geq \pi(1-\gamma)(1-C\varepsilon) |\nu_\varepsilon^2|(B_\sigma(x_i)) |\log \varepsilon| - C.$$

If $\liminf_{\varepsilon \rightarrow 0} |\nu_\varepsilon^2|(B_\sigma(x_i)) > |d_i|$, then in fact $\liminf_{\varepsilon \rightarrow 0} |\nu_\varepsilon^2|(B_\sigma(x_i)) \geq |d_i| + 1$ (because ν_ε^2 is integer-valued) and hence the Γ -liminf inequality (ii) follows, provided that we choose γ such that $(1-\gamma)(|d_i| + 1) > |d_i|$. Otherwise, we have that $|\nu_\varepsilon^2|(B_\sigma(x_i)) = |d_i|$ for ε small enough. Then, we can write

$$\nu_\varepsilon^2 = \sum_{j=1}^k p_j^\varepsilon \delta_{y_j^\varepsilon},$$

where the numbers $p_j^\varepsilon \in \mathbb{Z}$ all have the same sign and satisfy $\sum_j p_j^\varepsilon = d_i$, and $y_j^\varepsilon \rightarrow x_i$ as $\varepsilon \rightarrow 0$. By taking ε small enough, we can assume that $y_j^\varepsilon \in B_{\sigma/2}(x_i)$ for all j .

We fix a positive number $\eta > 0$ apply Corollary 4.7 with $s = s_\varepsilon^3 := \eta \#(\mathcal{B}_\varepsilon)^{-1}$. (We choose η small enough that $s_\varepsilon^3 \leq C R_* \#(\mathcal{B}_\varepsilon)^{-1}$, so the assumptions of Corollary (4.7) are satisfied.) We find a collection of balls $\mathcal{B}_\varepsilon^3 := \mathcal{B}_\varepsilon(s_\varepsilon^3)$ that satisfies $\text{spt}(\nu_\varepsilon^2) \subseteq \text{spt } \mathcal{B}_\varepsilon^3$,

$$(4.21) \quad \sum_{B \in \mathcal{B}_\varepsilon^3} \text{rad}(B) \leq C\eta$$

$$(4.22) \quad \frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(B \setminus \text{spt } \mathcal{B}_\varepsilon)}^2 \geq |\nu_\varepsilon(B)| \left(\pi(1-C\varepsilon) \log \frac{\eta}{\varepsilon \#(\mathcal{B}_\varepsilon)} - C \right)$$

Thanks to (4.21) and the fact that $\text{spt}(\nu_\varepsilon^2) \subseteq \text{spt } \mathcal{B}_\varepsilon^3$, $\text{dist}(\text{spt } \nu_\varepsilon^2, \partial B_\sigma(x_i)) \geq \sigma/2$, we can choose η so small that $B \subseteq B_\sigma(x_i)$ for any $B \in \mathcal{B}_\varepsilon^3$. Then, using also the fact that all the p_j^ε have the same sign and sum up to d_i , we see that $\sum_{B \in \mathcal{B}_\varepsilon^3} |\nu_\varepsilon^2(B)| = |d_i|$ and hence, by (4.22),

$$\frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(B_\sigma(x_i) \setminus \text{spt } \mathcal{B}_\varepsilon)}^2 \geq \pi |d_i| (1-C\varepsilon) \log \frac{\eta}{\varepsilon \#(\mathcal{B}_\varepsilon)} - C.$$

On the other hand, Lemma 4.4 implies that

$$\frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(\text{spt } \mathcal{B}_\varepsilon)}^2 \geq \sum_{T \in \mathcal{T}_\varepsilon : P(T) \cap S_\varepsilon \neq \emptyset} \frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(P(T))}^2 \geq \beta \#(\mathcal{B}_\varepsilon).$$

Thus, we have

$$\begin{aligned} \frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(B_\sigma(x_i))}^2 &\geq \pi |d_i| (1 - C\varepsilon) |\log \varepsilon| - \pi |d_i| (1 - C\varepsilon) \log \frac{\#(\mathcal{B}_\varepsilon)}{\eta} + \beta \#(\mathcal{B}_\varepsilon) - C \\ &\geq \pi |d_i| |\log \varepsilon| - C. \end{aligned} \quad \square$$

Proof of (iii) — Γ -limsup. Fix $\mu = \sum_{i=1}^K d_i \delta_{x_i} \in X$, and suppose that $d_i \neq 0$ for any i . By a diagonal argument, it suffices to show the following: for any δ and any countable subsequence of $\varepsilon \searrow 0$, there exists a (non-relabelled) subsequence such that

$$(4.23) \quad \limsup_{\varepsilon \rightarrow 0} \frac{|\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(M)}^2}{2|\log \varepsilon|} \leq \pi |\mu|(M) + \delta.$$

Let us fix $\delta > 0$ and the countable subsequence of ε . We also fix a small parameter $0 < \sigma \leq \sigma_0(\mu)$. For $i \in \{1, \dots, K\}$ and $j \in \{1, \dots, |d_i|\}$, we let $z_{i,j}^\varepsilon := \varepsilon^\sigma \exp(2\pi i j |d_i|^{-1}) \in \mathbb{C}$ (where i is the imaginary unit). By taking ε small enough, we can assume that $|z_{i,j}^\varepsilon| < \sigma$. We define a map $\tilde{\mathbf{u}}_i^\varepsilon: B_\sigma \subseteq \mathbb{C} \rightarrow \mathbb{C}$ by

$$(4.24) \quad \tilde{\mathbf{u}}_i^\varepsilon(z) := \prod_{j=1}^{d_i} \frac{z - z_{i,j}^\varepsilon}{|z - z_{i,j}^\varepsilon|} \quad \text{if } d_i > 0, \quad \tilde{\mathbf{u}}_i^\varepsilon(z) := \prod_{j=1}^{-d_i} \frac{\overline{z - z_{i,j}^\varepsilon}}{|z - z_{i,j}^\varepsilon|} \quad \text{otherwise.}$$

Using normal coordinates $\varphi_i: B_\sigma \subseteq \mathbb{C} \rightarrow M$ such that $\varphi_i(0) = x_i$, we can transport $\tilde{\mathbf{u}}_i^\varepsilon$ to a vector field on $B_\sigma(x_i) \subseteq M$, i.e. we define $\mathbf{u}_i^\varepsilon(\varphi_i(z)) := \langle d\varphi_i(z), \tilde{\mathbf{u}}_i^\varepsilon(z) \rangle$ for $z \in B_\sigma \subseteq \mathbb{C}$. Since $\sum_i \text{ind}(\mathbf{u}_i^\varepsilon, B_\sigma(x_i)) = \sum_i d_i = \chi(N)$, we find a smooth vector field \mathbf{u} on $M_\sigma := M \setminus \cup_i B_\sigma(x_i)$ that satisfies $\mathbf{u} = \mathbf{u}_i^\varepsilon$ on $\partial B_\sigma(x_i)$ for each i . We define $\mathbf{u}^\varepsilon := \mathbf{u}_i^\varepsilon$ on $B_\sigma(x_i)$ and $\mathbf{u}^\varepsilon := \mathbf{u}$ on M_σ . The tangent field $\mathbf{u}^\varepsilon \in W^{1,1}(M, \mathbb{R}^3)$ is smooth except at the points $x_{i,j}^\varepsilon := \varphi_i(z_{i,j}^\varepsilon) \rightarrow x_i$ and hence, by Lemma 3.5,

$$(4.25) \quad \frac{1}{2\pi} (\star d\mathbf{j}(\mathbf{u}^\varepsilon) - G) = \sum_{i,j} \text{sign}(d_i) \delta_{x_{i,j}^\varepsilon} \xrightarrow{\text{flat}} \sum_i d_i \delta_{x_i} = \mu \quad \text{as } \varepsilon \rightarrow 0.$$

Let \mathbf{v}_ε be the discrete field defined by $\mathbf{v}_\varepsilon(i) := \mathbf{u}^\varepsilon(i)$ for $i \in \mathcal{T}_\varepsilon^0$, and let $\mathbf{w}_\varepsilon := \hat{P}_\varepsilon^{-1} \circ \hat{\mathbf{v}}_\varepsilon$. We have

$$|\nabla \mathbf{w}_\varepsilon| \stackrel{(H_3)}{\leq} C |\nabla \hat{\mathbf{v}}_\varepsilon| \leq C |\nabla \mathbf{u}^\varepsilon|,$$

where the last inequality follows by basic properties of the affine interpolant. Setting $D_i^\varepsilon := B_\sigma(x_i) \setminus \cup_j B_\varepsilon(x_{i,j}^\varepsilon)$, and using (3.3) and Lemma 3.1, we obtain

$$\begin{aligned} \frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(M)}^2 &= \frac{1}{2} \sum_{i=1}^K |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(D_i^\varepsilon)}^2 + \frac{1}{2} \sum_{i,j} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(B_\varepsilon(x_{i,j}^\varepsilon))}^2 + \frac{1}{2} |\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(M_\sigma)}^2 \\ &\leq \frac{1+C\varepsilon}{2} \sum_{i=1}^K \int_{D_i^\varepsilon} |\nabla \mathbf{u}^\varepsilon|^2 \, dS + \sum_{i=1}^K \text{Lip}(\mathbf{w}_\varepsilon)^2 \mathcal{H}^2(B_\varepsilon(x_i)) + \frac{1+C\varepsilon}{2} \int_{M_\sigma} |\nabla \mathbf{u}|^2 \, dS \\ &\stackrel{(3.5)}{\leq} \frac{1+C\varepsilon}{2} \sum_{i=1}^K \int_{D_i^\varepsilon} |\nabla \mathbf{u}^\varepsilon|^2 \, dS + C_\sigma, \end{aligned}$$

where C_σ is a positive constant, depending on σ . The integral of $|\nabla \mathbf{v}|^2$ on each D_i^ε can be evaluated using (4.24) and the fact that $\text{Lip}(\varphi_i|_{B_\sigma(x_i)}) \leq 1 + C\sigma$:

$$\frac{1+C\varepsilon}{2} \int_{D_i^\varepsilon} |\nabla \mathbf{u}^\varepsilon|^2 \, dS \leq (\pi(1+C\sigma)|d_i| + C\sigma|d_i|^2) |\log \varepsilon| + C_\sigma,$$

whence

$$\limsup_{\varepsilon \rightarrow 0} \frac{|\mathbf{w}_\varepsilon|_{W_\varepsilon^{1,2}(M)}^2}{2|\log \varepsilon|} \leq \pi(1 + C\sigma)|\mu|(M) + C\sigma(|\mu|(M))^2$$

and, choosing σ so small that $C\sigma|\mu|(M) + C|\mu|(M)^2 \leq \delta$, (4.23) follows.

To conclude the proof, we only need to show that $\nu_\varepsilon(\mathbf{v}_\varepsilon)$ flat-converges to μ . Using (H_1) , the definition of affine interpolant, and the fact that

$$|\nabla \mathbf{u}^\varepsilon(x)| \leq \frac{C_\sigma}{\text{dist}(x, \{x_{i,j}^\varepsilon\})} \quad \text{for } x \in B_\sigma(x_i)$$

(as a consequence of (4.24)), one finds positive numbers λ, ε_1 such that, for any $0 < \varepsilon \leq \varepsilon_1$, there holds $|\mathbf{w}_\varepsilon| \geq 1/2$ on $A_\varepsilon := M \setminus \cup_{i,j} B_{\lambda\varepsilon}(x_{i,j}^\varepsilon)$. Thanks to Lemma 3.3, this implies $|\mathbf{u}_\varepsilon| \geq 1/4$ if ε is small enough, where \mathbf{u}_ε is the field defined by (3.23). Then, using (3.25) and (4.14), we obtain that $\nu_\varepsilon(\mathbf{v}_\varepsilon)[B] = 0$ if $B \subseteq A_\varepsilon$. On the other hand, we also have $\mu[B] = 0$ if $B \subseteq A_\varepsilon$, due to (4.25). Thus, for any Lipschitz function φ on M such that $\sup|\varphi| + \text{Lip}(\varphi) \leq 1$, there holds

$$\begin{aligned} \langle \nu_\varepsilon(\mathbf{v}_\varepsilon) - \mu, \varphi \rangle &= \sum_{i,j} \int_{B_{\lambda\varepsilon}(x_{i,j}^\varepsilon)} \varphi \, d(\nu_\varepsilon(\mathbf{v}_\varepsilon) - \mu) \\ (4.26) \quad &= \sum_{i,j} \int_{B_{\lambda\varepsilon}(x_{i,j}^\varepsilon)} (\varphi - \varphi(x_i)) \, d(\nu_\varepsilon(\mathbf{v}_\varepsilon) - \mu) \leq C\lambda\varepsilon(|\nu_\varepsilon(\mathbf{v}_\varepsilon)| + |\mu|)(M), \end{aligned}$$

and Lemma 4.5 implies

$$(4.27) \quad |\nu_\varepsilon(\mathbf{v}_\varepsilon)|(M) \leq C \# \{T \in \mathcal{T}_\varepsilon : P(T) \setminus A_\varepsilon \neq \emptyset\} \stackrel{(H_1)}{\leq} C.$$

Combining (4.26) and (4.27), we conclude that $\|\nu(\mathbf{v}_\varepsilon) - \mu\|_{\text{flat}} \leq C\lambda\varepsilon$. \square

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